This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

WENTWORTH-SMITH MATHEMATICAL SERIES

ACADEMIC ALGEBRA

BY

GEORGE WENTWORTH

AND

DAVID EUGENE SMITH

GINN AND COMPANY
BOSTON · NEW YORK · CHICAGO · LONDON
ATLANTA · DALLAS · COLUMBUS · SAN FRANCISCO

319.8

From math. dept.
2-24-1923
For Grad. H 3

The Athenæum Press
GINN AND COMPANY · PROPRIETORS · BOSTON · U.S.A.

PREFACE

In the preparation of this work the authors have consulted the courses of study in general use in the various states and leading cities of this country, and have considered with great care the syllabi and suggested curricula prepared by the various important associations of teachers of mathematics. They have also studied the papers recently set by the principal examination boards and have taken the judgment of a large number of prominent teachers as to the best selection and arrangement of topics for the first and second year's work in high-school algebra. As a result of this careful investigation, extending over a considerable period of time, they have prepared a work covering the topics generally agreed upon as suited to the pupil's needs and as fitting him for admission to our best colleges.

The plan adopted is at the same time new and conservative. The first chapter sets before the pupil some of the important uses of algebra, and recognizes the fact that the most important feature that he will meet to-day is the equation applied to the formula. It is the formula that the artisan first meets in his trade journal, that the mechanic needs in reading his manuals, and that the business man will use if he requires algebra at all. It is here that the modern function idea is best brought to the attention of the learner, and here he finds the natural connection between the mensuration that he has studied and the new science that he is beginning. The equation and the formula constitute, therefore, the best introduction to the subject.

Thereafter the work proceeds by important topics, a sufficient number of carefully graded drill problems being given with each to allow the teacher to select and change the material

from year to year. The applied problems include as many as possible of the kind met in real life, these necessarily being supplemented by others that have their chief value in the interest they arouse and the drill they give. The topics are those that are found in all standard courses of study extending through Progressions and the Binomial Theorem, and include a short chapter on the practical use of Logarithms. The chapter on Ratio and Proportion has been so simplified as to permit of its being placed where it naturally belongs, directly after fractional equations, thus allowing it to precede the work on similar figures in geometry, an arrangement that is generally recommended at the present time.

The authors have attempted to set forth, in simple language, the modern idea of function, without carrying the work to an unwarranted extreme. They have presented the graph in the same spirit, introducing it gradually and with the definite purpose of leading the pupil to recognize, through visual aids, the nature of negative numbers and the number and nature of the roots of equations.

Recognizing the importance of the equation, they have kept this topic before the student in all the chapters, and, in general, they have removed unnecessary barriers between topics whenever this could safely be done.

Particular attention is called to the Cumulative Review in the appendix. This furnishes the opportunity for a careful review of all the preceding work at the end of each chapter, and the value of the plan will be apparent to every teacher.

The authors wish to express their thanks to the many teachers who have assisted them either by their valuable suggestions or by their care in the reading of the proof. Any suggestions for further improvement of the work will be gratefully received.

GEORGE WENTWORTH
DAVID EUGENE SMITH

CONTENTS

TO THE TEACHER

The purpose of the Introduction (Chapter I) to the Wentworth-Smith Algebras, as in the case of most other modern textbooks, is to put the students in the right attitude of mind toward the subject and to lead them to see that algebra has something to do with everyday life. Therefore this Introduction makes clear at the outset many of the uses and applications of algebra. The formula, especially as applied to the equation, is the basis of the work, for it is the formula that the artisan meets first in his trade journal, that the mechanic needs in reading his manual, and that the business man will use if he requires algebra at all.

The exercises in this Introduction need not, however, necessarily be assigned to the class. The teacher may, if desired, utilize them as the basis for talks to the pupils about the new subject upon which they are just entering, and as illustrating the practical value of the equation and the formula in everyday life. An opportunity is thus afforded to show the pupils that algebra is a subject of practical value, thus avoiding the discouragement incident to the older and more formal introduction to this study. In case this plan is followed, the formal assignment of exercises may begin with Chapter II.

The modern demand for making use of the function in algebra is recognized on page 46. This topic is presented in such a simple form that any beginner can master it without the slightest difficulty. Any teacher who wishes to emphasize the subject is thus given an opportunity so to do. On the other hand, since the subsequent work does not, except in two or three examples which may be omitted, depend upon the function notation, the latter need not be given.

It should also be stated, in general, that all good textbooks furnish more material than is needed for the work of almost any class, the purpose being to afford an opportunity for varying the assignments from year to year.

ACADEMIC ALGEBRA

CHAPTER I

INTRODUCTION. CERTAIN USES OF ALGEBRA

1. Nature of Arithmetic. In arithmetic we usually represent numbers by figures. If the page of a book is seven inches long and four inches wide, we represent these dimensions by 7 in. and 4 in. respectively. If we wish to find the area, we say that

$$\text{area} = 4 \times 7 \text{ sq. in.} = 28 \text{ sq. in.}$$

2. Nature of Algebra. We might state the process of finding the area of the page as follows:

$$\text{area} = \text{width} \times \text{length},$$

meaning that the number of square inches of area equals the product of the number of inches of width and the number of inches of length. We do this in algebra, often using initial letters, thus:

$$a = w \times l.$$

Usually in algebra, however, we do not indicate multiplication by the sign ×. We write wl for $w \times l$, and hence we have

$$a = wl.$$

Arithmetically stated: *The number of units of area of a rectangle equals the product of the number of units of width and the number of units of length.*

Algebraically stated: $a = wl$.

One use of algebra, therefore, is the brief statement of the rules of arithmetic.

3. Arithmetical and Algebraic Statements. A comparison of arithmetical and algebraic statements is seen in the following:

Arithmetic	*Algebra*
If 1 book costs \$2, 3 books will cost 3 × \$2.	If 1 book costs d dollars, n books will cost nd dollars.
The cost of any number of books equals the cost of one book multiplied by the given number of books.	If c represents the total cost, then $c = nd$.
If a train travels 35 mi. an hour, in 4 hr. it will travel 4 × 35 mi., or 140 mi.	If a train travels m miles an hour, in h hours it will travel hm miles.
To find the distance traveled by a train in any number of hours, multiply the number of miles per hour by the number of hours.	If d represents the total distance, then $d = hm$.

4. Formula. A rule stated algebraically, in letters, is called a *formula.*

For example, $c = nd$, and $d = hm$, given above, are formulas.

Algebra enables us to express many of the rules of arithmetic very briefly by formulas.

None of the formulas should be memorized unless it is so stated.

In arithmetic we learn how to find the volume of an ordinary box. We say that the volume is equal to the product of the length, width, and height. In algebra, using initials as before, we represent this by the formula

$$v = lwh.$$

In this case, if $l = 3$, $w = 2$, and $h = 1$, we have

$$v = 3 \times 2 \times 1 = 6.$$

In writing algebraic forms it is not customary to express denominations like feet and inches. In arithmetic we would say that if $l = 3$ in., $w = 2$ in., and $h = 1$ in., v would equal $3 \times 2 \times 1$ cu. in., or 6 cu in.; but in algebra we omit the inches and cubic inches.

If $l = 4$, $w = 3$, and $h = 2$, we have, as before,

$$v = 4 \times 3 \times 2 = 24.$$

We therefore say that the volume of the box is 24 cu. in.

Exercise 1. Formulas

Examples 1 to 10, oral — Examples 11 to 13, written

1. If a rectangle is 8 in. long and 4 in. wide, how many square inches of area does it contain?

2. If a rectangle has a base of 12 in. and a height of 4 in., what is its area?

3. If a rectangle has a base b inches and a height h inches, what is its area?

4. If a rectangle is l inches long and w inches wide, how many square inches of area does it contain?

5. If 1 yd. of velvet costs $2, what will 9 yd. cost? If 1 yd. costs d dollars, what will y yards cost?

6. At the rate of 3 mi. an hour, how far will a man walk in 2 hr.? At the rate of m miles an hour, how far will he walk in h hours?

7. Read from the formula $a = mn$ the rule for finding the area of a rectangular field m rods wide and n rods long.

8. Read from the formula $v = pqr$ the rule for finding the volume of a rectangular box p inches long, q inches wide, and r inches deep.

9. Read from the formula $c = nd$ the rule for finding the cost (c) of a number of things when the cost of each (d) is given.

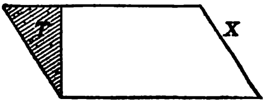
10. Read from the formula $n = c \div d$ the rule for finding the number of articles purchased when the cost of all (c) and the cost of each (d) are given.

11. Write a formula for the average cost (d) of each of n things when they cost c dollars in all.

12. Write a formula for the cost (c) of f feet of iron pipe at n cents a foot. Write the formula so as to express the result in cents; in dollars.

13. If n is any integer, is $2n$ an even number or an odd one? Why? What is the value of $2n$ when $n = 197$? 276? 997?

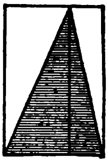
5. Formula for the Parallelogram. It is usually shown in arithmetic that the area of a parallelogram, like the area of a rectangle, equals the product of its base and height. We may therefore express this in algebraic form thus: $a = bh$.



For the triangle T may be cut off and placed at X, so as to make a rectangle of area bh.

If $b = 5$ and $h = 3$, then $a = bh = 5 \times 3 = 15$. If b and h represent inches, then a represents square inches. When we speak of the product of two lines we mean the product of their numerical values.

6. Formula for the Triangle. It is also shown in arithmetic that the area of a triangle equals half the product of its base and height. This is expressed in algebraic form thus: $a = \frac{1}{2}bh$.



This triangle may be cut as here shown so that it is seen to be half of the rectangle of base b and height h.

If $b = 7$ and $h = 10\frac{1}{2}$, then $a = \frac{1}{2}bh = \frac{1}{2}$ of $7 \times 10\frac{1}{2} = 36\frac{3}{4}$.

Exercise 2. The Parallelogram and Triangle

Examples 1 to 6, oral — Examples 7 to 11, written

1. If $b = 4$ and $h = 3$, what is the value of bh? of $\frac{1}{2}bh$?

2. If $b = 60$ and $h = 5$, what is the value of bh? of $\frac{1}{2}bh$?

Given $a = bh$, find the value of a when:

3. $b = 60$, $h = 11$.

4. $b = 45$, $h = 10$.

5. $b = 20$, $h = 12$.

6. $b = 22$, $h = 10$.

Given $a = \frac{1}{2}bh$, find the value of a when:

7. $b = 48$, $h = 25$.

8. $b = 36$, $h = 19\frac{1}{2}$.

9. $b = 24.8$, $h = 4.75$.

10. $b = 63.2$, $h = 19.65$.

11. A playground is l feet long and w feet wide. Find the area in square feet. What is the number of square feet when $l = 124$ and $w = 62\frac{1}{2}$?

7. Symbols. The following symbols of operation are among the ones that are most commonly used:

	Arithmetic	*Algebra*
Addition	$4 + 3$	$a + b$.
Subtraction	$4 - 3$	$a - b$.
Multiplication	4×3	$a \times b$, $a \cdot b$, or ab.
Division	$4 \div 3$	$a \div b$, $a : b$, or $\frac{a}{b}$.

In algebra the fraction form is the most common one for expressing the division of one quantity by another.

Second power (square) 5^2 means 5×5; a^2 means aa.

The second power, or square, of a number means the product arising from taking the number twice as a factor.

Third power (cube) 5^3 means $5 \times 5 \times 5$; a^3 means aaa.

The third power (cube) of a number means the product arising from taking the number three times as a factor, and so for other powers.

Square root	$\sqrt{4}$ means the square root of 4, or one of the two equal factors of 4.	$\sqrt{a}$ means the square root of a, or one of the two equal factors of a.

If $a = 3$, then $\sqrt{a}$ can be expressed only approximately as a decimal fraction, and similarly for $a = 5$, $a = 7$, and so on.

Cube root	$\sqrt[3]{27}$ means the cube root of 27, or one of the three equal factors of 27.	$\sqrt[3]{b}$ means the cube root of b, or one of the three equal factors of b.

It is customary to use the symbol $\therefore$ for "therefore."

8. Monomial. An algebraic expression in which the parts are not separated by the signs + or − is called a *monomial.*

Thus a, $3ab$, a^2, and $\sqrt{a}$ are monomials. By an algebraic expression we mean any expression in which letters are used to represent some or all of the numbers, like $a + b$, $3a$, $2n + 1$.

9. Polynomial. An algebraic expression consisting of two or more monomials is called a *polynomial.*

Thus $a + b$, $3a - 4b$, $2n + 1$, and $a - b + c$ are polynomials.

10. Terms of a Polynomial. The monomials that make up a polynomial are called the *terms* of the polynomial.

A polynomial is called a *binomial* if it has only two terms, and a *trinomial* if it has only three terms.

The terms of the binomial $a + 3b$ are a and $3b$.

The terms of the trinomial $a + 3b + c$ are a, $3b$, and c.

11. Symbols of Aggregation. Symbols that indicate that certain terms are to be treated as one number or one quantity are called *symbols of aggregation.*

The most common of these are the parentheses, brackets, and bar. Others will be given when needed. Thus $2 \times (5 + 7)$, $2 \times [5 + 7]$, or $2 \times \overline{5 + 7}$ means that 5 and 7 are to be added before we multiply by 2; each is read "twice the sum of 5 and 7," and equals 24. The expression $\sqrt{3 + 6}$, having the bar extending from the root sign over $3 + 6$, means that 3 and 6 are to be added before we extract the square root.

12. Order of Operations. In an algebraic expression the operations are performed in the following order, unless some symbols of aggregation direct otherwise:

1. *Powers and roots.*

2. *Multiplications and divisions, and these are taken in the order in which they occur.*

3. *Finally, additions and subtractions, and these may be taken in the order in which they occur or in any other order.*

Thus $\sqrt{4^3}$ means that we first cube 4, the result being 64; then take the square root of 64. It will be seen that we get the same answer if we first extract the square root of 4 and then cube the result.

In the case of $a^2b + c \div d$ we first square a; we then take the product of a^2 and b; we then divide c by d; and finally we add the results. Thus if $a = 2$, $b = 3$, $c = 10$, and $d = 5$, we have

$$2^2 \times 3 + 10 \div 5 = 4 \times 3 + 10 \div 5 = 12 + 2 = 14.$$

Similarly, we have the following:

$2\sqrt{4} + 6 \div 2 \times 3 = 2 \times 2 + 6 \div 2 \times 3 = 4 + 3 \times 3 = 4 + 9 = 13$;

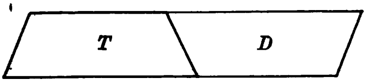
$5 \times 5 - 2^2 \times 3 + 12 \div 2 + 2 = 25 - 4 \times 3 + 6 + 2 = 21$;

$\sqrt{3 + 6} + 15 \div 5 - 2 + 7 \times 6 = 3 + 3 - 2 + 42 = 46$;

$24 \div 12 \times 2^2 - 5 + 2 = 24 \div 12 \times 4 - 5 + 2 = 2 \times 4 - 5 + 2 = 5.$

13. Formula for the Trapezoid. It is usually shown in arithmetic that the area of a trapezoid equals half the product of the sum of the two parallel sides multiplied by the height.

For a trapezoid D, equal to the given trapezoid T, may be turned over and put down by the side of T, as here shown. The whole figure, or twice the trapezoid, then equals a parallelogram whose base is the sum of the parallel sides of the trapezoid. The trapezoid is therefore half this parallelogram.



We indicate this in algebraic form as follows:

$$a = \frac{1}{2}(b + b')\,h,$$

where a is the area, b and b' (" b prime ") are the two parallel sides, usually called the *bases*, and h is the height.

The parentheses show that b and b' are to be added before being multiplied by h.

Thus if $b = 6$, $b' = 5$, and $h = 4$, we have $a = \frac{1}{2}(b + b')\,h = \frac{1}{2}(6 + 5) \times 4 = \frac{1}{2}$ of $11 \times 4 = 22$.

Exercise 3. The Trapezoid

Examples 1 to 7, oral — Examples 8 to 13, written

1. Find the value of $b + b'$, when $b = 7$, $b' = 9$; when $b = 7$, $b' = 9\frac{1}{2}$; when $b = 7\frac{3}{4}$, $b' = 9\frac{1}{2}$.

2. Find the value of $(b + b')\,h$, when $b = 4$, $b' = 5$, $h = 3$.

Given $a = \frac{1}{2}(b + b')\,h$, *find the value of* a *when:*

3. $b = 6$, $b' = 4$, $h = 3$.
4. $b = 9$, $b' = 5$, $h = 7$.
5. $b = 10$, $b' = 8$, $h = 6$.
6. $b = 13$, $b' = 7$, $h = 7$.
7. $b = 19$, $b' = 5$, $h = 4$.
8. $b = 24$, $b' = 9.5$, $h = 7.2$.
9. $b = 34$, $b' = 16$, $h = 19$.
10. $b = 38$, $b' = 9.8$, $h = 8.6$.
11. $b = 5.9$, $b' = 3.4$, $h = 1\frac{1}{2}$.
12. $b = 4\frac{3}{4}$, $b' = 3\frac{1}{4}$, $h = 2\frac{1}{2}$.

13. A playground is in the form of a trapezoid, with bases x and y, and with height z. What is the area? How many square rods are there in the playground if $x = 30$ rd., $y = 26$ rd., and $z = 24$ rd.? if $x = 34$ rd., $y = 28$ rd., and $z = 30$ rd.?

14. Evaluation of an Expression. The substitution of numerical values for the letters in an algebraic expression, and the reduction of the result to simplest form, is called the *evaluation* of the expression.

Thus to evaluate $2n$ for $n = 5$, substitute 5 for n and we have $2 \times 5 = 10$.

Exercise 4. Evaluation of Expressions

Examples 1 to 7, oral — Examples 8 to 20, written

1. What are the terms of the binomial $a + b$? Evaluate it for $a = 1$ and $b = 2$.

2. What are the terms of the trinomial $x + y + z^2$? Evaluate it for $x = 1$, $y = 1$, $z = 1$.

3. In the expression $4 \times (5 + 2)$, which operation is performed first? Which next? What is the result? What is the value of $4 \times 5 + 2$? What is the value of $4 \times 5 + 2 \times 3 \div 3$?

Evaluate the following:

4. $a = b^2$; $b = 3$.
5. $a = bh$; $b = 12$, $h = 4$.
6. $a = bh$; $b = 22$, $h = 6$.
7. $a = \frac{1}{2}bh$; $b = 22$, $h = 10$.
8. $2x^2 + 3x + 1$; $x = 10$.
9. $3x^3 + 4x^2 + 2x + 3$; $x = 10$.
10. $a + 6b$; $a = 7$, $b = 1$.
11. $(a + b)^2$; $a = 1$, $b = 2$.
12. $(a - b)^2$; $a = 7$, $b = 2$; also $a = 10$, $b = 5$.
13. $a^2 - 2ab + b^2$; $a = 7$, $b = 2$; also $a = 10$, $b = 5$.
14. $a^2 - b^2$; $a = 5$, $b = 3$; also $a = 2$, $b = 1$.
15. $(a + b)(a - b)$; $a = 5$, $b = 3$; also $a = 2$, $b = 1$.
16. $(a + b)^3$; $a = 4$, $b = 3$; also $a = 2$, $b = 10$.
17. $a^3 + 3a^2b + 3ab^2 + b^3$; $a = 4$, $b = 3$; also $a = 3$, $b = 4$.
18. $(a - b)^3$; $a = 23$, $b = 4$; also $a = 49$, $b = 43$.
19. $\sqrt{a^2 + 2ab + b^2}$; $a = 4$, $b = 2$; also $a = 5$, $b = 3$; $a = 7$, $b = 1$.
20. $\sqrt{a^2 - 2ab + b^2}$; $a = 5$, $b = 1$; also $a = 5$, $b = 3$; $a = 6$, $b = 5$.

Exercise 5. Writing Algebraic Expressions

Examples 1 to 14, oral — Examples 15 to 22, written

1. Read $4 + 7$; $a + b$; $3a + b$; $3a + 5b$.

2. Read $9 - 5$; $a - b$; $7a - b$; $7a - 4b$.

3. Read 2×7; $a \times b$; ab; $3ab$; $a(b + c)$; $x(a + b + c)$.

4. Read $9 \div 3$; $a \div 3$; $a \div b$; $3a \div b$; $9a \div b$; $9a \div 7$.

5. Read $2n$; $2n + 1$; $2n - 1$. What is the value of each when $n = 3$? when $n = 5$? when $n = 10$?

6. Read $\frac{1}{2}n$; $\frac{1}{3}n$; $\frac{2}{3}n$; $\frac{3}{4}n$. What is the value of each when $n = 12$? when $n = 24$? when $n = 10$?

7. How many feet in a yard? in 5 yd.? in n yards?

8. How many inches in a foot? in 2 ft.? in n feet?

9. How many ounces in a pound? in 10 lb.? in n pounds?

10. If 1 yd. of cloth costs 20¢, what will 3 yd. cost? n yards?

11. If 1 lb. of tea costs \$½, what will 4 lb. cost? n pounds?

12. Read 2 ft.; 2 yd.; $2n$; $2x$; 2 mi.; $2m$; 2 A.; $2a$.

13. Read 3 ft. 2 in.; $3x + 2y$; $3a + 2b$; \$3 + 2¢.

14. Read 7 ft. − 2 in.; 7 yd. − 2 ft.; \$7 − 2¢; $7x - 2y$.

15. Write the sum of 2 and y; of x and y; of $2x$ and y.

16. Indicate the subtraction of y from 7; of y from x.

17. Write the products: 2 times x; 2 times y; x times y.

18. Write the quotient of x divided by 2, both as a fraction and with some other sign of division.

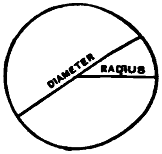
19. In the same two ways write the quotient of $2x$ divided by 3; of $2x$ divided by y; of $2x$ divided by $3y$.

20. Write the sum of $5x^2$, $7x$, and 2. What is its value when $x = 10$? What is its value when $x = 20$?

21. Write the sum of $7x^3$, $2x^2$, $3x$, and 6. What is its value when $x = 10$? What is its value when $x = 20$?

22. Write the sum of $2x^4$, $3x^3$, $3x^2$, $7x$, and 5. What is its value when $x = 10$? What is its value when $x = 1$?

15. Formula for the Circumference. It is usually shown in arithmetic that the circumference of a circle equals nearly 3.1416 × the diameter. In mathematics the number 3.1416 —, or 3.14159 +, which is nearly $3\frac{1}{7}$, is represented by the Greek letter π (pronounced *pī*). We may therefore express this law as follows:



$$c = \pi d,$$

where c stands for circumference, d for diameter, and π for nearly $3\frac{1}{7}$, or nearly 3.1416.

Since the diameter equals twice the radius, we may write $2r$ for d, and have $c = \pi \times 2r$, or $c = 2\pi r$.

Thus if $d = 7$, and we take $3\frac{1}{7}$ or $\frac{22}{7}$ as the value of π, we have

$$c = \pi d = 3\tfrac{1}{7} \times 7 = 22.$$

If $r = 5$, we have $c = 2\pi r = 2 \times 3\frac{1}{7} \times 5 = 31\frac{3}{7}$.

Exercise 6. The Circumference

Examples 1 to 4, oral — Examples 5 to 21, written

1. If $\pi = 3\frac{1}{7}$ and $d = 7$, find the value of πd.

2. If $\pi = 3\frac{1}{7}$ and $r = 3\frac{1}{2}$, find the value of $2r$; of $2\pi r$.

Given $c = \pi d = 2\pi r$, and taking $\pi = 3\frac{1}{7}$, find c when:

3. $d = 14$. 5. $d = 3.5$. 7. $r = 77$. 9. $r = 6.3$.
4. $d = 21$. 6. $d = 4.9$. 8. $r = 3.5$. 10. $r = 9.1$.

Taking $\pi = 3.1416$, find c when:

11. $d = 10$. 13. $d = 50$. 15. $r = 30$. 17. $r = 2\frac{1}{2}$.
12. $d = 20$. 14. $d = 25$. 16. $r = 40$. 18. $r = 7.5$.

19. A boy measures the diameter of his bicycle wheel and finds it to be 28 in. What is the circumference?

20. A workman measures the diameter of a steel shaft and finds it to be $3\frac{7}{16}$ in. What is the circumference?

21. If you describe a circle with a radius of 12 in., how many inches will there be in the circumference?

16. Formula for the Area of a Circle. It is usually shown in arithmetic that

1. The area of a circle equals half the product of the circumference and radius; that is, $a = \frac{1}{2} cr$. a = π 25

2. The area of a circle equals π times the square of the radius; that is, $a = \pi r^2$.

Thus if $r = 10$, we have $a = \pi r^2 = 3.1416 \times 10^2 = 314.16$, taking 3.1416 for π. If r represents the number of feet, the area is in square feet; if r represents the number of inches, the area is in square inches.

Exercise 7. Area of a Circle

Examples 1 to 4, oral — Examples 5 to 23, written

1. If $r = 1$ and $c = 6\frac{2}{7}$, what is the value of cr? of $\frac{1}{2} cr$?

2. If $r = 10$ and $c = 63$, what is the value of cr? of $\frac{1}{2} cr$?

3. If $r = 1$ and $\pi = 3\frac{1}{7}$, what is the value of r^2? of πr^2?

4. If $r = 2$ and $\pi = 3\frac{1}{7}$, what is the value of r^2? of πr^2?

Given $a = \frac{1}{2} cr$, find a when:

5. $r = 5$, $c = 31.416$.
6. $r = 10$, $c = 62.832$.
7. $r = 20$, $c = 125.664$.
8. $r = 2\frac{1}{2}$, $c = 15.708$.
9. $r = 25$, $c = 157.08$.
10. $r = 50$, $c = 314.16$.

Given $a = \pi r^2$, and taking $\pi = 3\frac{1}{7}$, find a when:

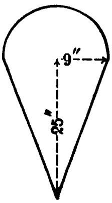
11. $r = 7$.
12. $r = 3\frac{1}{2}$.
13. $r = 14$.
14. $r = 2.1$.
15. $r = 2.8$.
16. $r = 35$.
17. $r = 4.9$.
18. $r = 7.7$.

Given $a = \pi r^2$, and taking $\pi = 3.1416$, find a when:

19. $r = 5$. **20.** $r = 10$. **21.** $r = 20$.

22. How many square feet in the area of a circle whose radius is 2 ft.? (Take $\pi = 3.1416$.)

23. How many square inches are there in the area of this kite? (Take $\pi = 3\frac{1}{7}$. Also find the result when 3.1416 is taken for π.)



Exercise 8. Evaluation of Expressions

Examples 1 to 9, oral — Examples 10 to 33, written

1. Evaluate $a+b$ for $a=7$, $b=5$; for $a=9$, $b=11$.
2. Evaluate $2a+b$ for $a=1$, $b=5$; for $a=5$, $b=5$.
3. Evaluate $2a+3b$ for $a=1$, $b=1$; for $a=2$, $b=1$.
4. Evaluate $a-b$ for $a=7$, $b=5$; for $a=9$, $b=7$.
5. Evaluate $4a-b$ for $a=1$, $b=1$; for $a=2$, $b=1$.
6. Evaluate ab for $a=2$, $b=3$; for $a=\frac{1}{2}$, $b=12$.
7. Evaluate $a \div b$ for $a=6$, $b=2$; for $a=9$, $b=3$.
8. Evaluate $2n$ for $n=1$; for $n=7$; for $n=12$; for $n=47$.
9. Evaluate $2n+1$ for $n=1$; for $n=7$; for $n=59$.

Copy and write the values:

10. $3n$; $n=7, 5, 9, 39$.
11. $3n+1$; $n=7, 5, 11, 77$.
12. $3n-1$; $n=5, 7, 16, 28$.
13. $5n$; $n=17, 39, 179$.
14. $5n+1$; $n=26, 37, 79$.
15. $5n-1$; $n=43, 72, 67$.
16. $6n$; $n=32, 48, 79, 87$.
17. $6n+1$; $n=75, 97, 128$.
18. $7n+1$; $n=36, 42, 88$.
19. $7n-1$; $n=44, 78, 96$.

Taking $\pi = 3.1416$, evaluate the following:

20. $2\pi r$; $r=15$.
21. πr^2; $r=12$.
22. $\pi r^2 h$; $r=1$, $h=3$.
23. $4\pi r^2$; $r=12$.
24. $\frac{4}{3}\pi r^3$; $r=9$.
25. $2\pi r$; $r=37$.
26. πr^2; $r=75$.
27. $\pi r^2 h$; $r=2$, $h=7$.
28. $4\pi r^2$; $r=7\frac{1}{2}$.
29. $\frac{4}{3}\pi r^3$; $r=12$.
30. If $x=21$, find the values of $x+7$; $x-7$; $7x$; $x \div 7$.
31. If $x=51$, find the values of $96+x$; $96-x$; $96x$.
32. If $x=75$, find the values of $x+25$; $x-25$; $25x$.
33. A rectangular subway is to be made w feet wide and h feet high. The earth weighs p pounds per cubic foot. How many tons of earth will be removed in l feet of length? Evaluate the result for $w=35$, $h=16$, $l=1000$, $p=70$.

17. Passing from Words to Symbols. One of the great values of algebra is its power of expressing words by symbols. This we have already seen in several formulas.

For example, instead of writing "the square of some number added to four times the number, and the sum diminished by 7," we may write $x^2 + 4x - 7$, or $n^2 + 4n - 7$.

To express algebraically five times any number, we may write $5n$, $5N$, or $5x$, etc. Capital letters are often used, particularly in formulas.

Exercise 9. From Words to Symbols

Examples 1 to 3, oral — Examples 4 to 30, written

1. Express algebraically three less than twice any number.

2. Express algebraically the sum of any two numbers; twice the sum of any two numbers.

3. Express algebraically the difference of any two numbers; three times the difference; half of the difference.

Write in figures, using the appropriate signs:

4. 6 times the sum of 4 and 5.

5. 5 times the difference between 17 and 8.

6. $\frac{2}{3}$ of the sum of 9, 15, 18, 27, 33, and 36.

Express algebraically:

7. n times the sum of a and 3; of a and $3b$.

8. n times the difference between 19 and 7; between a and b

9. 7 less than twice a; than half of a; than a times b.

10. y more than the quotient of a divided by 5.

Express the cost of:

11. 3 things at 5¢ each; n things at 5¢ each.

12. n things at c cents each; at $3c$ cents each.

13. n things at 60¢ a dozen; at c cents a dozen.

14. n things when a of them cost c cents.

15. x things at 4 for 25¢; at n for c cents; at x for 3¢.

State the distance traveled in:

16. 3 hr. at 3 mi. per hour; h hours at m miles per hour.

17. T hours at 45 mi. per hour; at m miles per hour.

18. t hours at 1 mi. in 2 min.; at M miles in H hours.

19. m minutes at 4 mi. per hour; at a miles in b hours.

Express algebraically the following statements:

20. The square of the sum of two numbers; the sum of the squares of the numbers.

21. The square of the difference of two numbers; the difference of the squares of the numbers.

22. The sum of two numbers is equal to their difference plus twice the smaller number.

23. The sum of two numbers plus their difference is equal to twice the larger number.

24. The sum of the squares of two numbers is equal to twice the product of the numbers, plus some other number.

25. The weight in pounds of a pine beam with a square end is 40 times the length multiplied by the square of the thickness, the foot being the unit.

26. The weight in pounds of a cube of granite is 168 times the third power of its edge, the foot being the unit.

27. The square of the sum of two numbers equals the sum of their squares, together with twice their product.

28. The square of the difference of two numbers equals the sum of their squares, less twice their product.

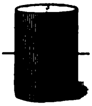
29. The product of the sum of two numbers and the difference of the numbers equals the difference of the squares of the numbers.

30. The sum of two numbers multiplied by the larger number equals the square of the larger number together with the product of the two numbers.

Exercise 10. Formulas

Examples 1 to 12, oral — Examples 13 to 33, written

1. It is usually shown in arithmetic that the volume (v) of a cylinder equals the product of the base (b) and height (h). Express this in a formula.



2. It is also shown that the volume equals the product of the height and π times the square of the radius of the base. Express this in a formula.

Express the following statements in formulas:

3. The volume (V) of a cube equals the third power of an edge (e).

4. The entire surface (S) of a cube equals six times the second power of an edge (e).

5. The lateral (or side) area of a cylinder equals the product of the circumference and height. (Use L for lateral area, c for circumference, and h for height.)

6. The lateral area of a cylinder equals 3.1416 times the diameter multiplied by the height.

7. The lateral area of a cylinder equals twice 3.1416 times the radius multiplied by the height. (Use r for radius, and in general use the initial letter for a word in a formula.)

8. The volume of a prism equals the product of the base and height.

9. The volume of a cylinder equals the product of the base and height.

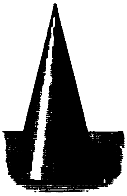
10. The volume of a cylinder divided by the base equals the height.

11. The volume of a pyramid equals one third the product of the base and height.

12. The volume of a pyramid divided by one third the height equals the base.

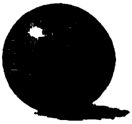
13. The volume of a cone equals one third the product of 3.1416 times the square of the radius of the base multiplied by the height.

14. The formula for the lateral surface of a cone in terms of the circumference (c) of the base and the distance (k) from the vertex to the circumference of the base is $l = \frac{1}{2} kc$. Write this as an ordinary sentence.



15. The formula for the lateral surface of a cone in terms of the radius of the base and the distance from the vertex to the circumference of the base is $l = \pi kr$. Write this as an ordinary sentence.

16. The formula for the surface of a sphere is $s = 4 \pi r^2$. Write this as an ordinary sentence.

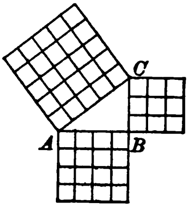


17. How many square inches in the surface of a sphere of which the radius is 7 in.?

18. The formula for the volume of a sphere is $v = \frac{4}{3} \pi r^3$. Write this as an ordinary sentence.

19. How many cubic inches in the volume of a sphere of which the radius is 7 in.?

20. The square on the hypotenuse of a right triangle is, as seen in the following figure, equal to the sum of the squares on the other two sides. Letting $AB = x$, $BC = y$, and $AC = z$, we have $z^2 = x^2 + y^2$, or $z = \sqrt{x^2 + y^2}$. Suppose $x = 4$ and $y = 3$, find the value of z.



21. Given $z = \sqrt{x^2 + y^2}$, find the value of z when $x = 24$ and $y = 32$.

22. In the same formula find the value of z when $x = 36$ and $y = 27$.

23. In the same formula find the value of z when $x = 33$ and $y = 44$.

24. In the same formula find the value of z when $x = 48$ and $y = 36$; when $x = 5.1$ and $y = 6.8$; when $x = 0.51$ and $y = 0.68$.

25. If an automobile has a constant rate of 12 mi. an hour, how far will it go in $3\frac{1}{2}$ hr.? Write a formula for d, the distance that it will go in t hours, at v miles per hour.

26. What is the volume of a box 12 in. long, $8\frac{1}{2}$ in. wide, and 6 in. deep? Write a formula for v, the volume of a box that is l inches long, w inches wide, and d inches deep.

27. The formula for the radius of a circle in terms of the circumference is $r = \frac{c}{2\pi}$. Write this as an ordinary sentence.

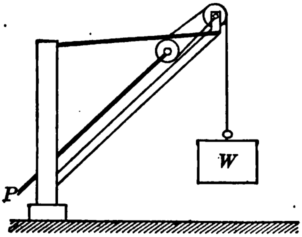
28. If it is c inches around an iron pipe, the diameter is $\frac{c}{\pi}$ inches; that is, $d = \frac{c}{\pi}$. Write this as an ordinary sentence.

29. The circumference of a water pipe is 22 in. Using the formula in Ex. 28, and taking $3\frac{1}{7}$ as the value of π, find the diameter.

30. The circumference of a wheel is 13.2 ft. Find the diameter. Find the radius.

31. A man who runs a stationary engine for hoisting iron reads in a book about engines that if s is the area of the outside shell of his boiler and h is the heating surface, then $h = \frac{2}{3}s$. What does this mean? What is the value of h if $s = 120\frac{5}{16}$ sq. ft.?

32. A foreman of a shop has a hoist like this for lifting heavy weights. He reads in a trade journal that in order to lift a weight W, the power P must be such that $P = \frac{1}{2}W$. What power must his engine apply to the cable so as to lift a weight of 18,000 lb.?



33. The volume of a cylinder being represented by $v = \pi r^2 h$, what will the water in a cylindrical water tank weigh when $r = 7$, $h = 10$, $\pi = 3\frac{1}{7}$, the dimensions being in feet and the weight of 1 cu. ft. of water being $62\frac{1}{2}$ lb.?

Exercise 11. Formulas and Problems

Examples 1 to 4, oral — Examples 5 to 11, written

1. State in your own language the rule for finding the area of a rectangle. State this as a formula.

2. State the rule and also a formula for finding the area of a triangle.

3. State the rule and also a formula for finding the area of a parallelogram.

4. State the rule and also a formula for finding the circumference of a circle.

Write the formulas for the following rules, using initial letters, and evaluate for the given values of the letters:

5. The volume of a prism equals the product of the base and height. (Let $b = 27$, $h = 3\frac{1}{2}$.)

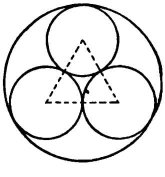
6. The volume of a cube equals the third power of the edge. (Let $e = 21.3$.)

7. The volume of a pyramid equals one third the product of the base and height. (Let $b = 48$, $h = 35$.)

8. The volume of a cylinder equals approximately 3.1416 times the square of the radius, multiplied by the height. (Let $r = 1$, $h = 2$.)

9. The volume of a cone equals one third of π times the square of the radius of the base, multiplied by the height. (Let $r = 10$, $h = 30$, $\pi = 3.1416$.)

10. The volume of a sphere equals four thirds of 3.1416 times the cube of the radius. (Let $r = 2$. Also solve for $r = 10$.)



11. If the radius of the large circle in this figure is R and the radius of each of the small circles is r, write a formula for the area of that part of the large circle not covered by the small circles (see § 16). Evaluate for $R = 10.77$, $r = 5$.

Exercise 12. Commercial Formulas

Examples 1 to 4, oral — Examples 5 to 14, written

1. At 6% a year, what is the interest on $200 for 1 yr.?

2. If r is the rate for 1 yr., what is the interest on p dollars for 1 yr.?

3. At 6% a year, what is the interest on $200 for 2½ yr.?

4. If r is the rate for 1 yr., what is the interest on p dollars for t years?

5. If the list price of some goods is $575 and a discount of 10% is allowed, what is the discount? What is the net price?

6. If the list price of some goods is l dollars, and the rate of discount is r, what is the discount? Write a formula for n, the net price.

7. If some stock is selling 7% below par, what will be the cost of $1500 worth, par value, brokerage not considered?

8. If some stock is selling at $r\%$ below par, what will be the cost of p dollars' worth, par value, brokerage not considered?

9. What is the amount of principal and interest of $1700 for 2 yr. at 5%?

10. What is the amount of principal and interest of p dollars for t years, the rate of interest being r?

11. A trade price list gives the cost in dollars per foot of sewer pipe of diameter d inches, as follows: $c = 0.004\,d^2 + 0.14$. Find the cost of 300 ft. of 16-inch pipe.

12. From the formula of Ex. 11, find the cost of half a mile of 18-inch pipe.

13. From the formula of Ex. 11, find the cost of 30 rd. of 8½-inch pipe.

14. If the amount of d dollars on interest for n years at $r\%$ is $d(1 + nr\%)$, what is the amount of $750 for 3½ yr. at 5%?

Exercise 13. Shop Formulas

Examples 1 to 4, oral — Examples 5 to 16, written

1. Given $b = 7$, $h = 9$, what is the value of bh?

2. Given $b = 12$, $h = 20$, what is the value of $\frac{1}{2}bh$?

3. Given $r = 10$, what is the value of r^2? of r^3?

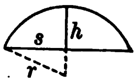
4. Given $r = 1$, $\pi = 3\frac{1}{7}$, what is the value of πr^2?

5. The diameter of a steel shaft of circumference c is $\frac{c}{\pi}$. Evaluate to two decimal places for $c = 10$, $\pi = 3\frac{1}{7}$.

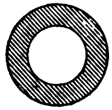
6. A bar of metal has for cross section an equilateral triangle each side of which is s. The area of the cross section is given by the formula $a = \frac{1}{4}s^2\sqrt{3}$. Find to two decimal places the area of the cross section when $s = 7$.

7. The foreman of a shop reads in his book of instruction that the safe load (l) in pounds that can be hoisted by a rope c inches in circumference is given by the formula $l = 100c^2$. How many pounds can he safely allow for a rope that is $2\frac{1}{4}$ in. in circumference?

8. A carpenter wishes to put up a circular arch of height h and span $2s$. It is necessary to find the radius of the circle so that he may make his pattern. He knows that $r = \frac{s^2 + h^2}{2h}$. Find the radius, given $h = 2$, $s = 4$.



9. The area of a circle with radius a being πa^2, and of one with radius b being πb^2, the area of the ring between the two circumferences is $\pi a^2 - \pi b^2$. What is the area of the ring if $a = 7$, $b = 4$, and $\pi = 3\frac{1}{7}$?

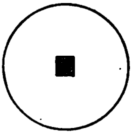


10. A workman needs to find the area of the metal in a cross section of iron pipe of which the exterior diameter is 10 in. and the interior diameter 9 in. Find this from the formula $\pi a^2 - \pi b^2$, where a and b are radii (semidiameters), and $\pi = 3\frac{1}{7}$.

11. In the shop a workman has to cut a circular hole in a circular iron plate. The diameter of the plate is 12 in. and the diameter of the hole is 7 in. The formula for the area of a circle in terms of the diameter is $\frac{1}{4}\pi d^2$. Required the area of the circular plate left after cutting the hole. Take $\pi = 3\frac{1}{7}$.

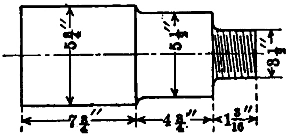
12. On the roof of a shop is a water tank in which the surface of the water is 40 ft. above the ground, and from the tank a pipe leads down through the shop. The owner wishes to find how strong this pipe must be so that it will not burst. He reads in a book on water pipes that the formula for the pressure of water in pounds on every square inch, the surface of the water being h feet above the ground, is denoted by the formula $p = 62\frac{1}{2}h \div 144$. What is the pressure on the pipe at the level of the ground?

13. Given $v = \pi r^2 h$ as the formula for the volume of a cylinder, find the weight of a steel shaft 6 ft. long and 2 in. in diameter, a cubic inch of steel weighing 0.283 lb.



14. This circular plate has a radius of r inches. A square hole s inches on a side is cut in the plate. What is the area of the remaining part?

15. A machinist is making a "crank pin" (a kind of bolt) for an engine, according to this drawing. He considers it as weighing the same as three steel cylinders having the diameters and lengths in inches as here shown, where $7\frac{3}{4}''$ means $7\frac{3}{4}$ in. He has this formula for the weight (w) in pounds of a steel cylinder where d is the diameter and l is the length in inches: $w = 0.07\,\pi d^2 l$. Taking $\pi = 3\frac{1}{7}$, find the weight of the pin.



16. The volume of a cylinder being represented by $v = \pi r^2 h$, what will the water in a cylindrical water tank weigh when $r = 9$ ft., $h = 10$ ft., $\pi = 3\frac{1}{7}$, and the weight of 1 cu. ft. of water is $62\frac{1}{2}$ lb.?

18. Equation. An expression of equality between two numbers or quantities is called an *equation.*

For example, $x + 5 = 7$. This means that some number increased by 5 equals 7. Evidently this number is 2.

In this equation x is usually called the *unknown quantity.* It is really a number to be determined. Unknown quantities are usually represented by x, y, or z, or else by initial letters, such as p for pounds or d for dollars.

In this equation $x + 5$ is called the *first member* and 7 is called the *second member.*

Exercise 14. Using Subtraction

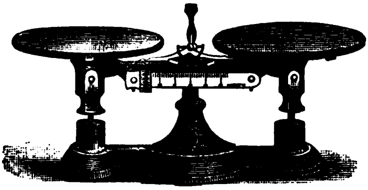
Examples 1 to 10, oral — Examples 11 to 18, written

1. What number increased by 4 equals 7? In the equation $n + 4 = 7$ what is the number n?

2. What weight increased by 5 lb. equals 11 lb.? In the equation $p + 5 = 11$ what is the value of p?

3. What number increased by 7 equals 15? What is the value of x in the equation $x + 7 = 15$?

4. If these scales balance when we place $(x + 6)$ oz. on one side and 15 oz. on the other side, how much will be left on each of the sides if we take 6 oz. from each? What is the value of x?



5. If we take 10 oz. from one side, how much must we take from the other side to keep the scales balanced?

6. If equals are subtracted from equals, what can be said of the remainders?

Find the value of x in the following:

7. $x + 9 = 21$.
8. $x + 19 = 21$.
9. $x + 19 = 49$.
10. $x + 29 = 79$.
11. $x + 1.7 = 9.1$.
12. $x + 5.7 = 9.2$.
13. $x + 0.76 = 0.93$.
14. $x + 0.77 = 0.95$.
15. $x + 9\frac{1}{2} = 10\frac{1}{8}$.
16. $x + 3\frac{1}{4} = 4\frac{1}{2}$.
17. $x + 5\frac{2}{3} = 7\frac{3}{4}$.
18. $x + 9\frac{7}{8} = 10\frac{3}{4}$.

Exercise 15. Using Division

Examples 1 to 19 oral — Examples 20 to 29. written

1. If twice a certain number equals 16, what is the number? If $2n = 16$, what is the value of n?

2. If 5 times a certain number equals 35, what is the number? If $5n = 35$, what is the value of n?

3. At 6¢ each, how many oranges can I buy for 48¢? If $6n = 48$, what is the value of n?

4. At 12¢ a yard, how many yards of ribbon can I buy for 48¢? If $12r = 48$, what is the value of r?

5. At 40¢ a dozen, how many dozen eggs can I buy for $1.60? If $40e = 160$, what is the value of e?

6. If I put equal weights on the two sides of some scales like those shown on page 22, then $\frac{1}{5}$ of the weight on one side will just balance what part of the weight on the other side?

7. If equals are divided by equals, what can be said of the quotients?

Find the value of x in the following:

8. $2x = 6$.
9. $3x = 9$.
10. $3x = 6$.
11. $4x = 8$.
12. $5x = 5$.
13. $6x = 6$.
14. $7x = 21$.
15. $9x = 63$.
16. $7x = 91$.
17. $4x = 72$.
18. $5x = 75$.
19. $6x = 72$.
20. $13x = 117$.
21. $17x = 119$.
22. $16x = 128$.
23. $1.9x = 11.4$.
24. $2.3x = 25.3$.
25. $0.27x = 0.324$.

26. What number multiplied by 15 equals 240?

27. What number multiplied by 17 equals 221?

28. A man paid $780 for a certain number of cattle at $65 a head. How many did he buy?

29. A dealer paid $10,325 for some automobiles at an average price of $1475 each. How many did he buy?

Exercise 16. Using Addition

Examples 1 to 13, oral — Examples 14 to 30, written

1. What must be added to 3 to make 8? to 8 − 5 to make 8? to $x - 5$ to make x? to $n - 5$ to make n?

2. If $x - 5 = 20$, what must be added to these equals to make the first member x? What does the second member then become? What is the value of x?

3. If $n - 7 = 31$, what must be added to these equals to make the first member n? What is the value of n?

4. If $n - 9 = 35$, how will you proceed to find the value of n? What is the value of n?

5. If 6 is subtracted from a certain number, the result is 12. What is the number?

6. If equals are added to equals, what can be said of the results?

Find the value of x in the following:

7. $x - 1 = 7$.
8. $x - 2 = 7$.
9. $x - 4 = 6$.
10. $x - 5 = 8$.
11. $x - 6 = 9$.
12. $x - 7 = 3$.
13. $x - 8 = 5$.
14. $x - 27 = 72$.
15. $x - 34 = 62$.
16. $x - 71 = 89$.
17. $x - 8.7 = 8.7$.
18. $x - 4.9 = 5.7$.
19. $x - 0.78 = 0.35$.
20. $x - 0.87 = 0.63$.
21. $x - 2\frac{1}{2} = 3\frac{1}{2}$.
22. $x - 2\frac{1}{4} = 5\frac{3}{4}$.
23. $x - 6\frac{2}{3} = 7\frac{2}{3}$.
24. $x - 4\frac{7}{8} = 5\frac{3}{4}$.
25. $x - 3\frac{4}{5} = 8\frac{1}{2}$.
26. $x - 6\frac{2}{3} = 4\frac{1}{4}$.
27. $x - 9\frac{3}{4} = 8\frac{3}{5}$.

28. If 59 is subtracted from a certain number, the result is 59. What is the number? Prove it.

29. If 279 is subtracted from a certain number, the result is 643. What is the number? Prove it.

30. A book has a certain number of pages. After we have read 169 pages there are 138 more to be read. How many pages are there in the book?

Exercise 17. Using Multiplication

Examples 1 to 12, oral — Examples 13 to 28, written

1. By what must $\frac{1}{2}$ in. be multiplied to make 1 in.?

2. By what must $\frac{1}{2}x$ be multiplied to make x?

3. By what must $\frac{3}{4}$ be multiplied to make 3?

4. By what must $\frac{3}{4}x$ be multiplied to make x?

5. By what must both members of the equation $\frac{x}{4} = 7$ be multiplied to give the value of x? What is the value of x?

6. If $\frac{x}{3} = 9$, what is the value of x? Prove it.

7. If equals are multiplied by equals, what can be said of the results?

Find the value of x in the following:

8. $\frac{x}{2} = 9.$
9. $\frac{x}{3} = 7.$
10. $\frac{x}{4} = 8.$
11. $\frac{x}{5} = 5.$
12. $\frac{x}{6} = 4.$
13. $\frac{x}{11} = 10.$
14. $\frac{x}{17} = 16.$
15. $\frac{x}{13} = 14.$
16. $\frac{x}{15} = 12\frac{1}{3}.$
17. $\frac{x}{16} = 14\frac{3}{4}.$
18. $\frac{x}{2.5} = 26.$
19. $\frac{x}{2.7} = 32.$
20. $\frac{x}{4.1} = 3.9.$
21. $\frac{x}{5.2} = 1.7.$
22. $\frac{x}{0.61} = 0.75.$

23. What number divided by 26 equals 26?

24. What number divided by $3\frac{1}{2}$ equals $4\frac{3}{4}$?

25. If $\frac{1}{8}$ of a certain distance is 29.4 ft., what is the distance?

26. If $\frac{1}{32}$ of the thickness of a steel plate is $\frac{1}{8}$ in., what is the thickness of the plate?

27. If $\frac{1}{48}$ of the diameter of a steel rod is $\frac{1}{96}$ in., what is the diameter of the rod?

28. If 1% of the cost of a house is $47.50, what is the cost of the house?

19. Axioms. A general statement admitted to be true without proof is called an *axiom.*

The axioms needed in the beginning of algebra are six in number. Others will be added as required.

1. *If equals are added to equals, the results are equal.*
2. *If equals are subtracted from equals, the results are equal.*
3. *If equals are multiplied by equals, the results are equal.*
4. *If equals are divided by equals, the results are equal.*
5. *Like powers or like roots of equals are equal.*
6. *Quantities equal to the same quantity are equal to each other.*

Exercise 18. Use of the Axioms

Examples 1 to 7, oral — Examples 8 to 19, written

1. If $2a = 16$, what does a equal? What axiom is used?

2. If $2a + 1 = 9$, what does $2a$ equal? What axiom is used? Then what does a equal? What axiom is used?

3. If $3a - 1 = 14$, what does $3a$ equal? Then what does a equal? What two axioms are used?

4. If $\frac{1}{3}a + 1 = 7$, what does $\frac{1}{3}a$ equal? Then what does a equal? What two axioms are used?

5. If $\frac{1}{5}a - 1 = 4$, what does $\frac{1}{5}a$ equal? Then what does a equal? What two axioms are used?

6. If $\frac{2}{3}a = 12$, what does $\frac{1}{3}a$ equal? Then what does a equal? What two axioms are used?

7. If we divide both members of the equation $\frac{2}{3}a = 12$ by $\frac{2}{3}$, what is the result? What axiom is used?

Find the value of a in the following:

8. $3a = 111$.

9. $7a = 112$.

10. $8a = 112$.

11. $9a = 135$.

12. $3a + 1 = 112$.

13. $7a + 6 = 118$.

14. $8a + 9 = 121$.

15. $9a - 5 = 130$.

16. $\frac{2}{3}a = 70$.

17. $\frac{3}{4}a = 66$.

18. $\frac{4}{5}a = 72$.

19. $\frac{5}{8}a = 75$.

20. Algebra and Arithmetic compared. If to twice a certain number we add 7, the result is 33. Required the number.

Solution by Algebra

Let n stand for the number.
Then $2n$ is twice the number,
and $2n + 7 = 33$, as stated in the problem.
Therefore $2n = 26$, by subtracting 7 from equals,
and $n = 13$, by dividing equals by 2.
Check or *Proof.* $2 \times 13 + 7 = 33$. Therefore the work is correct.

Solution by Arithmetic

Because twice the number added to 7 equals 33, therefore if 7 is taken away from 33 there will remain twice the number. Therefore 26 is twice the number. Therefore once the number is half of 26, or 13.

$$\begin{array}{r} 33 \\ \underline{7} \\ 2\overline{)26} \\ 13 \end{array}$$

21. How to solve a Problem. From the above problem we see that the algebraic solution makes the reasoning clearer. To solve a problem by algebra,

(1) *Write a letter for the number sought;*
(2) *Use this letter in the statement of the problem;*
(3) *This will give an equation representing the problem;*
(4) *Find the value of the letter; that is, solve the equation.*

Exercise 19. Solution of Problems

Examples 1 to 6, oral — Examples 7 to 21, written

1. If we wish to find a certain number such that if 7 is added to 3 times this number the result is 40, by what shall we represent this number in an equation?

2. Then how shall we represent 3 times the number?

3. How shall we represent this added to 7?

4. How shall we express this sum as equal to 40?

5. What shall we do to these equals so as to leave n alone on one side of the equation?

6. How shall we check or prove that our result is correct?

7. If to a certain number we add 11, the result is 29. What is the number?

8. If to 7 times a certain number we add 3, the result is 45. What is the number?

9. If to 23 times a certain number we add 47, the result is 300. What is the number?

10. If from 19 times a certain number we subtract 17, the result is 173. What is the number?

11. If from 2.3 times a certain number we subtract 2.8, the result is 13.3. What is the number?

12. If from $3\frac{4}{5}$ times a certain number we subtract $2\frac{1}{5}$, the result is $16\frac{4}{5}$. What is the number?

13. If to twice a certain number we add $3\frac{7}{8}$, the result is $4\frac{1}{8}$. What is the number?

14. If from twice a certain number we subtract $3\frac{7}{8}$, the result is $4\frac{1}{8}$. What is the number?

15. The length of a room exceeds the width by 6 ft. The width is 18 ft. What is the length? What is the area of the floor?

16. Twice the width of a room is 8 ft. more than the length of the room. The width is 12 ft. What is the length?

17. If to 4 times the length of a room we add 6 ft., the result is 70 ft.; and if to 5 times the width of the room we add 10 ft., the result is also 70 ft. Find the length and width.

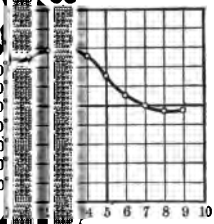
18. A biplane traveling at the rate of d miles a minute goes 8.4 mi. in 7 min. Find the value of d.

19. A train traveling at the rate of m miles an hour goes $112\frac{1}{2}$ mi. in $2\frac{1}{2}$ hr. Find the value of m.

20. If a man's annual income is increased by $600, the result is the same as twice his annual income decreased by $1800. What is his annual income?

21. If a man's annual income is increased by $100, the result is the same as three times his annual income decreased by $3900. What is his annual income?

[illegible] hours of the [illegible]



[illegible] nine hours.

[illegible]

[illegible] written

[illegible] ture at 4 P.M.?

[illegible]?

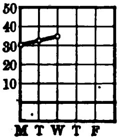
[illegible] M.?

[illegible] 50°?

[illegible]

[illegible] following:

[illegible] was 30 on [illegible]



[illegible]

Rule some paper and draw lines to show the following:

7. The season's record of a certain baseball team is as follows: To May 1 the per cent of games won to games played was 68.4%; to June 1 it was 67.6%; to July 1, 67.2%; to Aug. 1, 63.7%; to Sept. 1, 64.2%; to Oct. 1, 64.7%. The line is here shown. Draw a similar line for these per cents: 65%, 66%, 65.5%, 64.2%, 63.5%, 62%. Draw a similar line for these per cents: 54%, 48%, 39%, 45%, 52%, 55.5%.

8. The pressure on the steam gauge of a boiler varies as follows: 6 A.M. 120 lb., 7 A.M. 122 lb., 8 A.M. 132 lb., 9 A.M. 130 lb., 10 A.M. 126 lb., 11 A.M. 130 lb., noon 124 lb., 12.30 P.M. 120 lb., 1 P.M. 125 lb., 2 P.M. 130 lb., 3 P.M. 128 lb., 4 P.M. 126 lb., 5 P.M. 124 lb., 6 P.M. 112 lb.

9. A boy's height from the age of 5 to the age of 15, stated in inches, varied as follows: 5 yr., 42; 6 yr., 44; 7 yr., 46; 8 yr., 49; 9 yr., 52; 10 yr., 54; 11 yr., 56; 12 yr., 58; 13 yr., 61; 14 yr., 63; 15 yr., 68.

10. A girl's height from the age of 5 to the age of 15, stated in inches, varied as follows: 5 yr., 42; 6 yr., 44; 7 yr., 45; 8 yr., 48; 9 yr., 50; 10 yr., 52; 11 yr., 54; 12 yr., 58; 13 yr., 60; 14 yr., 63; 15 yr., 65.

11. The population of the United States, in millions, for various years was as follows: 1820, 10; 1830, 13; 1840, 17; 1850, 23; 1860, 31; 1870, 39; 1880, 50; 1890, 63; 1900, 76; 1910, 92. In such a case it is convenient to use 0.1 in. to represent 10 millions.

12. In order to determine the course of a tunnel a cross-section plan of a river bed had to be made from soundings taken at various distances from the left bank, the measurements being made in feet. Draw a plan of the river bed from the following data:

Distance from bank,	0, 10, 20, 30, 40, 45, 50, 51, 52, 55, 60.
Depth of river,	9, 12, 15, 25, 25, 28, 35, 37, 35, 20, 8.

23. Numbers below Zero. When it is necessary to distinguish between temperature below zero and temperature above zero, we write 10° above zero, $+10°$; and 10° below zero, $-10°$.

If the temperature is 20° above zero and it decreases 15°, it is then 5° above zero, or $+5°$. If it decreases 5° more, it is then 0°. If it decreases 5° more, it is 5° below zero, or $-5°$. If it decreases 20° more, it is 25° below zero, or $-25°$.

We therefore find a new meaning for the signs + and —. They not only indicate addition and subtraction (signs of *operation*), but they tell on which side of zero a number is (sign of *quality*).

24. Positive Numbers. The ordinary numbers which we use in arithmetic are called *positive numbers.*

Thus 3°, 3 in., $\frac{3}{4}$, $\sqrt{3}$, are all positive numbers. If we wish to make this fact emphatic, we may write them thus: $+3°$, $+3$ in., $+\frac{3}{4}$, $+\sqrt{3}$, but otherwise the + sign is unnecessary here. The expression $+3$ is read "positive 3" or "plus 3."

25. Negative Numbers. Numbers on the other side of zero from positive numbers are called *negative numbers.*

Thus $-3°$ is a negative number. If distance upwards, above the earth's surface, is called positive, distance below the surface is called negative, so that we may have $+10$ ft. and -10 ft. The expression -3 is read "negative 3" or "minus 3." We may think of zero as either positive or negative, since it divides the two classes of numbers.

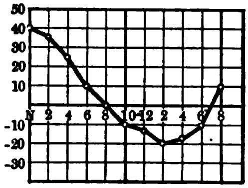
If the temperature is 20° below zero, or $-20°$, and it increases 7°, it is then 13° below zero, or $-13°$. If it increases 9° more, it is 4° below zero, or $-4°$. If it increases 4° more, it is then zero, or 0°. If it increases 32° more, it is then 32° above zero, or $+32°$.

26. Absolute Value. The numerical value of a quantity, without reference to its sign, is called its *absolute value.*

The absolute value of -4 is 4, and that of $-a$ is a.

27. Curve Tracing with Negative Numbers. If the temperature in St. Paul, on a winter's day, falls from + 40° at noon to – 20° at 2 A.M., and then rises again to + 10° at 8 A.M., as here shown, the curve tells us that it was below zero from 8 P.M. until about 7 A.M.

In this case we represent negative numbers below the horizontal line, which represents zero.



Exercise 21. Curve Tracing

Examples 1 to 7, oral — Examples 8 to 10, written

1. By the above curve, when was the temperature 35°?

2. At what times was it – 10°? + 10°? + 5°? – 5°?

3. What was the temperature at 6 A.M.? at 8 A.M.?

4. At what time did the temperature cease falling?

5. How much did it rise from 2 A.M. to 6 A.M.? to 8 A.M.?

6. What was the difference in temperature at 6 P.M. and 10 P.M.? at 8 P.M. and 2 A.M.? at 10 P.M. and 2 A.M.?

7. When the temperature is – 10°, how much must it rise to be 0°? to be + 10°? to be + 20°?

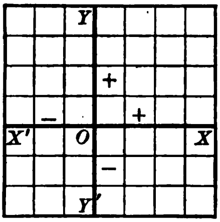
Trace the curves to show the following variations in temperature for the twenty-four hours:

8. Noon 33°, 2 P.M. 42°, 4 P.M. 34°, 6 P.M. 30°, 8 P.M. 22°, 10 P.M. – 8°, midnight – 8°, 2 A.M. – 10°, 4 A.M. – 5°, 6 A.M. 0°, 8 A.M. 12°, 10 A.M. 30°, noon 34°.

9. Noon 48°, 2 P.M. 45°, 4 P.M. 40°, 6 P.M. 25°, 8 P.M. 0°, 10 P.M. – 5°, midnight – 8°, 2 A.M. – 5°, 4 A.M. 0°, 6 A.M. 5°, 8 A.M. 10°, 10 A.M. 20°, noon 35°.

10. Noon 0°, 2 P.M. 5°, 4 P.M. 10°, 6 P.M. 8°, 8 P.M. 0°, 10 P.M. – 5°, midnight – 10°, 2 A.M. – 5°, 4 A.M. – 2°, 6 A.M. 10°, 8 A.M. 20°, 10 A.M. 32°, noon 45°.

28. Other Uses of Negative Numbers. If we call some special point on a line zero (0), we usually call distances to the right positive and distances to the left negative, just as we call distances up (as on the thermometer) positive and distances down negative. But because we usually call weight positive, we speak of the weight of a balloon (which pulls upwards) negative.



The following are some additional illustrations of negative numbers:

If a man is worth \$1000, we may say that he has + \$1000; but if he is \$1000 in debt, we may say that he is worth – \$1000.

If we have 25 on a score in a game, we have + 25; but if we are 25 worse off than nothing, we have – 25.

If we call latitude north of the equator positive, we may call south latitude negative.

If we call longitude west of Greenwich positive, we may call east longitude negative; and if we call west longitude negative, we should call east longitude positive.

If we call the motion of a piston rod of an engine positive when it is to the right, we may call it negative when it is to the left.

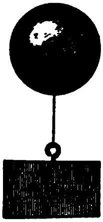
If we call downward pressure positive, we may speak of upward pressure as negative.

If we call distance above the earth's surface positive, we may call distance below the earth's surface negative.

We therefore see that negative numbers are just as real as positive numbers, for the temperature is just as real when the thermometer indicates that it is below zero as it is when the mercury rises above zero, and a man's debts are just as real as his capital.

In ancient times people used only whole numbers (integers). Other kinds of numbers were invented as they became necessary, and these are sometimes called artificial numbers. Artificial numbers like $\frac{3}{4}$, $\sqrt{3}$, and -3 all have their uses, as we have seen, not only in the theory of algebra but in its practical applications.

29. Adding Negative Numbers. If we tie to a 10-pound weight a toy balloon that pulls upward 1 pound, what will the two together weigh? From the answer to this question we find the following:



To add a positive number to a negative number, take the difference of their absolute values and prefix the sign of the numerically greater number.

Thus $+10$ lb. and -1 lb. are $+9$ lb.;
$+10$ lb. and -10 lb. are 0 lb.;
$+10$ lb. and -15 lb. are -5 lb.

Similarly, *to add a negative number to a negative number, take the sum of their absolute values and prefix the negative sign.*

Illustrate this, using two balloons, one pulling upward 5 lb. and the other pulling upward 6 lb.

Exercise 22. Addition

Examples 1 to 5, oral — Examples 6 to 36, written

1. What is the combined weight of $+25$ lb. and -5 lb.?

2. What is the combined weight of 30 lb. and -60 lb.?

3. A freight engine is switching in front of a station. If it runs 500 ft. to the right of the station ($+500$ ft.) and then backs 525 ft. (-525 ft.), how many feet is it from the station? (Add 500 and -525.)

4. In drilling a well the drill is raised $+8$ ft. above the surface of the ground. It is then dropped 16 ft. (-16 ft.). Where is it then with respect to the surface? (Add 8 and -16. A negative distance above the surface means below the surface.)

5. A boy is fishing in deep water with a line 22 ft. long. If the tip of the pole is $+6$ ft. above the water, how far is the sinker from the surface of the water, if it is 3 ft. from the hook? (Add 6, -22, and 3.)

Add as indicated:

6. $-275+316$.

7. $-486+531$.

8. $-279+603$.

9. $-481+296$.

10. $-370+198$.

11. $-760+436$.

12. $243+(-48)$.

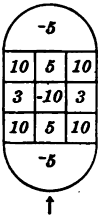
13. $300+(-97)$.

14. $421+(-84)$.

15. A man who was worth $4500 lost $1750 and then earned $900. How much was he then worth? (Add 4500, -1750, and 900.)

16. A man who was $450 in debt contracted another debt of $250. He then earned $1000. How much was he then worth?

17. A game is played by throwing bean bags in the direction of the arrow. Suppose the score stands -5, 5, 3, 10, -10, 5, 10, 10, 3, 3, -5, how much is the total score?



18. If this board without any weights at the ends just balances, and if I put 5 lb. at one end and 8 lb. at the other end, how much must I add to the 5 lb. to make it balance? Instead of adding to the 5 lb., how much must I add to the 8 lb.?

19. A boat that goes 14.1 mi. an hour in still water is going against a stream flowing 3.8 mi. an hour. What is the rate at which the boat will travel? (14.1 mi. and -3.8 mi. are how many miles?)

20. If a mine is opened 400 ft. above the base of a mountain and a shaft is sunk 750 ft., how much is the base of the shaft above or below the base of the mountain?

Add the following:

21.	**22.**	**23.**	**24.**	**25.**	**26.**
-62	75	60	-30	-90	-35
62	-80	-60	-40	-72	-75

27. Find the average noon temperature for the week in which the noon temperatures were 15°, 3°, 0°, $-7°$, $-20°$, 6°, 25°.

28. A mass of iron and wood is placed in a tank of water. The iron tends to sink the mass with a force of 20 lb., and the wood tends to buoy it up with a force of 16 lb. What does the mass weigh under water?

29. If my watch is 5 min. faster than the schoolroom clock, and the clock is 7 min. slower than the correct time, how near is my watch to the correct time?

30. In a tug of war one group of boys pulls to the north with a force of $256\frac{3}{4}$ lb., and the other group pulls to the south with a force of $252\frac{1}{2}$ lb. What is the resulting force?

31. On a rock that is 6 ft. 8 in. below the average level of the sea a lighthouse is built 81 ft. 4 in. high. How high is the top of the lighthouse above the average sea level?

32. At a point on a hill 329 ft. above a valley the shaft of a mine is sunk to a depth of 401 ft. How far is the bottom of the shaft below the valley?

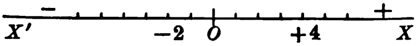
33. An aëroplane that can fly 58.2 mi. an hour in still air is flying against a wind that retards it 9.7 mi. an hour. At what rate does the aëroplane fly?

34. An elevator starting from the main floor of a high building runs up 9 stories, then down 10 stories, and then up 1 story. Where is it then? Express the solution by using positive and negative numbers.

35. A certain office building is 37 stories above the street and 4 stories below. An elevator starting at the street level ascends 28 stories, descends 30, ascends 35, descends 37, and ascends 26. Where is the elevator then? Represent the movements by a diagram, using positive and negative signs.

36. A man having $374.75 in the bank deposits $176.50 on Monday, checks out $482.60 on Tuesday, deposits $243.85 on Wednesday, and checks out $281.45 on Thursday. What is his balance? Add the positive numbers, then the negative numbers, and then the two sums.

30. Subtracting a Negative Number. If the temperature is $-10°$ at midnight and $+40°$ at noon, the difference in temperature is evidently 50°, for the mercury must rise 10° to reach 0°, and 40° more to reach $+40°$. Likewise, in this figure the difference between -2 and $+4$ is 6; for a point must move 2 spaces to get from -2 to 0, and 4 more to reach $+4$; that is, 6 must be added to -2 to make 4.



To subtract a negative number we may obtain the same result by adding a positive number with the same absolute value.

That is, $4-(-2)=4+2=6$. Likewise $-5-(-3)=-5+3=-2$. This is the usual rule for subtracting a negative number, but it is never necessary actually to change the sign of the subtrahend. If we subtract a smaller from a larger number, such as -10 from -4, the result is necessarily positive. If we subtract a larger from a smaller number the result is necessarily negative.

Exercise 23. Subtraction

Examples 1 to 8, oral — Examples 9 to 16, written

1. How much difference in price is there in selling a horse at $25 below cost or at $30 above cost?

2. The temperature on one morning was $+11°$, and the next morning $-7°$. What was the difference in temperature?

3. If there is a house for every number, how many houses would you pass in going from 48 East Washington Street to 17 West Washington Street, including both these houses?

4. Jefferson Street is 6 blocks east of Adams Street, and Monroe Street is 9 blocks west of Adams Street. Monroe Street is how many blocks west of Jefferson Street?

5. $4-(-3)$.
6. $5-(-7)$.
7. $6-(-3)$.
8. $7-(-7)$.
9. $0-(-47)$.
10. $36-(-49)$.
11. $59-(-59)$.
12. $78-(-96)$.
13. $13.7-(-27.8)$.
14. $16.5-(-43.4)$.
15. $37.8-(-96.8)$.
16. $52.7-(-88.9)$.

31. Subtracting a Positive Number. If the temperature is – 10° at midnight and falls 2° more during the next hour, it is then – 12°. That is, to subtract 2° from – 10° is the same as to add – 2° to – 10°. Therefore

In subtracting a positive number we may obtain the same result by adding a negative number with the same absolute value.

Hence $(+a) - (+b) = (+a) + (-b) = a - b$. It is not necessary to remember such a rule, for if we are taking a smaller from a larger number the result is positive, and in the contrary case it is negative.

Exercise 24. Subtraction

Examples 1 to 17, oral — Examples 18 to 29, written

1. How much is 10° – 5°? 0° – 5°? – 5° – 5°?

2. How much is \$20 – \$10? \$0 – \$10? \$5 – \$10?

3. How much is 15 ft. – 6 ft.? 0 ft. – 6 ft.? – 12 ft. – 6 ft.?

4. How much is 12¢ – 7¢? 7¢ – 7¢? 5¢ – 7¢?

5. How much is 25 lb. – 15 lb.? 15 lb. – 15 lb.? 5 lb. – 15 lb.?

6. 17 – 7.
7. 17 – 17.
8. 17 – 27.
9. 6 – 5.
10. 4 – 5.
11. – 4 – 5.
12. 7 – 3.
13. 1 – 3.
14. – 1 – 3.
15. – 4 – 7.
16. – 5 – 6.
17. – 8 – 3.

18. If a man has \$178 and incurs a debt of \$275, how much is he then worth?

19. If a man has – \$178 (that is, if he is \$178 in debt) and incurs a debt of \$275, how much is he then in debt?

20. 69 – 32.
21. 69 – 72.
22. – 69 – 32.
23. 48 – 75.
24. 52 – 86.
25. 73 – 91.
26. – 36 – 92.
27. – 49 – 76.
28. – 56 – 83.

29. If two trains pass each other in the opposite direction at 9.45 A.M., one going at the rate of 47.6 mi. an hour, and the other at the rate of 39.7 mi. an hour, how far apart will they be at 10.15 A.M. if these rates are maintained?

32. Multiplying and Dividing Negative Numbers. We multiply and divide negative numbers just as we multiply and divide positive numbers. If a man has $-\$5$ (is \$5 in debt), he will have $-\$10$ if he is twice as much in debt.

That is, $$2 \times (-\$5) = -\$10,$$

and $$-\$10 \div 2 = -\$5.$$

Exercise 25. Multiplication and Division

Examples 1 to 4, oral — Examples 5 to 20, written

1. If one balloon pulls up 400 lb. (weighs -400 lb.), what will be the upward pull of 3 such balloons? Represent the weight as a negative number.

2. If the thermometer indicates $-8°$, what is the temperature when it indicates half as much below zero? When it indicates twice as much below zero?

3. If a carrier pigeon can fly 38 mi. an hour in still air, at what rate will it fly against a 15-mile wind? against a wind that blows twice as fast? against a hurricane that blows three times as fast?

4. A checker of bales of cotton finds one bale 16 lb. short, a second bale twice as much short, and a third bale half as much short in weight. Express these shortages of weight in algebraic language.

5. A man's debts amounted last year to \$475. The year before they were 4 times as much. This year he has paid his debts and has \$825 in the bank. What is the difference between his financial standing year before last and now?

6. $3 \times (-86)$.
7. $4 \times (-49)$.
8. $6 \times (-73)$.
9. $7 \times (-89)$.
10. $8 \times (-96)$.
11. $1.7 \times (-3.2)$.
12. $2.8 \times (-4.9)$.
13. $-125 \div 25$.
14. $-493 \div 17$.
15. $-4.93 \div 1.7$.
16. $-43.4 \div 7$.
17. $-5.58 \div 9$.
18. $-7.92 \div 9$.
19. $-40.7 \div 3.7$.
20. $-5.39 \div 4.9$.

33. Multiplying by a Negative Number. We cannot pick up a book $2\frac{2}{3}$ times. Nevertheless we say that $2\frac{2}{3}$ times \$3 equals \$8. That is, we define what is meant by multiplying by $2\frac{2}{3}$, and then we use the word "times" just as we do with integers.

Similarly, we cannot pick up a book -2 times, but we may define what we mean by multiplying by -2, and then we may use the word "times" as we do with positive integers.

Because $3 \times (-2) = -6$, therefore -2×3 ought to equal -6. Therefore we define multiplication by a negative number to mean multiplication by a positive number having the same absolute value, the sign of the product being then changed.

Therefore

$$\begin{array}{rr} 2 \times (-3) = -6. & a \cdot (-b) = -ab. \\ -2 \times 3 = -6. & -a \cdot b = -ab. \\ -2 \times (-3) = 6. & -a \cdot (-b) = ab. \end{array}$$

If two numbers have like signs, their product is positive; if they have unlike signs, their product is negative.

Exercise 26. Multiplication

Examples 1 to 7, oral — Examples 8 to 13, written

1. If each of 3 men spends \$2 in a store, how much does the store receive? How much is $3 \times \$2$?

2. If each of 3 men steals \$2 from a store, how much does the store gain or lose? How much is $3 \times (-\$2)$?

3. If 3 men who would have spent \$2 each in a store are persuaded to trade elsewhere, how much does the store gain or lose in gross receipts? How much is $-3 \times \$2$?

4. If 3 men who would have stolen \$2 each from a store are arrested before the theft, how much does the store gain?

5. $3 \cdot (-9)$.
6. $-3 \cdot 9$.
7. $-3 \cdot (-9)$.
8. $-27 \cdot 63$
9. $-42 \cdot (-76)$.
10. $-29 \cdot (-38)$.
11. $-96 \cdot 83$.
12. $-2 \cdot (-3) \cdot (-7)$.
13. $-8 \cdot (-8) \cdot (-9)$.

34. Dividing by a Negative Number. Division being the inverse of multiplication, because

$$\begin{array}{llll} 2 \times 3 & = 6, \text{ therefore } & 6 \div 3 & = 2; \\ 2 \times (-3) & = -6, \text{ therefore } & -6 \div (-3) & = 2; \\ -2 \times 3 & = -6, \text{ therefore } & -6 \div 3 & = -2; \\ -2 \times (-3) & = 6, \text{ therefore } & 6 \div (-3) & = -2. \end{array}$$

If two numbers have like signs, their quotient is positive; if they have unlike signs, their quotient is negative.

Exercise 27. Division

Examples 1 to 16, oral — Examples 17 to 33, written

1. How much is $2 \cdot (-7)$? $-14 \div 2$? $-14 \div (-7)$?
2. How much is $-3 \cdot 8$? $-24 \div (-3)$? $-24 \div 8$?
3. How much is $-7 \cdot (-9)$? $63 \div (-7)$? $63 \div (-9)$?
4. How much is $-a \cdot (-b)$? $ab \div (-a)$? $ab \div (-b)$?
5. $25 \div (-5)$.
6. $-25 \div 5$.
7. $-25 \div (-5)$.
8. $-25 \div (-25)$.
9. $36 \div (-4)$.
10. $-36 \div 4$.
11. $-36 \div (-4)$.
12. $-36 \div (-9)$.
13. $56 \div (-8)$.
14. $56 \div (-7)$.
15. $-56 \div (-8)$.
16. $-56 \div (-7)$.

17. By what number must 17 be multiplied to make 544? to make – 544?

18. By what number must – 19 be multiplied to make – 399?

19. By what number must – 22 be multiplied to make 770? to make – 770?

20. $625 \div 25$.
21. $625 \div (-25)$.
22. $-625 \div 25$.
23. $-625 \div (-25)$.
24. $-675 \div (-25)$.
25. $3367 \div 37$.
26. $3367 \div (-37)$.
27. $-3367 \div (-37)$.
28. $-3367 \div (-91)$.
29. $34.3 \div 7$.
30. $34.3 \div (-7)$.
31. $-34.3 \div 7$.
32. $-34.3 \div (-7)$.
33. $-34.3 \div (-0.7)$.

35. The System of Integers. We now see that our system of integers extends indefinitely on both sides of zero, thus:

$$\cdots -6,\ -5,\ -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6 \cdots$$

36. The Properties of Zero. The following are the important properties of zero:

$$a + 0 = a, \qquad 0 - a = -a, \qquad 0 \cdot a = 0,$$
$$a - 0 = a, \qquad a \cdot 0 = 0, \qquad 0 \div a = 0.$$

The expression $a \div 0$ may be thought of for the present as having no meaning, division by zero not being allowed. It is considered later.

Exercise 28. Review

Examples 1 to 32, oral — Examples 33 to 44, written

1. $4 + (-3)$.
2. $6 + (-9)$.
3. $9 + (-3)$.
4. $1 + (-9)$.
5. $7 + (-3)$.
6. $-2 + 8 \times 2 \div 4$.
7. $-8 + 4 \div 2 \times 3$.
8. $-5 + 7 \times 3$.
9. $-5 - 9 \div 9$.
10. $-7 - 2 \times 4$.
11. $2 \times (-7) \times 3$.
12. $5 \times (-3) - 2$.
13. $8 \times (-6) + 6$.
14. $-5 \times 7 \div 4$.
15. -8×9.
16. $-7 \times (-8)$.
17. $8 \div (-2)$.
18. $-8 \div 2$.
19. $-6 \div 2$.
20. $-8 \div (-2)$.
21. $-7 \div (-1)$.

22. How much is $-3 \times (-3)$? $(-3)^2$? $(-7)^2$? $(-9)^2$?

23. How much is $9 - 2 \times (-2) \times (-2)$? $9 - (-2)^3$?

24. $(-1)^2$.
25. $(-1)^5$.
26. $(-2)^4$.
27. $(-2)^5$.
28. $(-5)^2$.
29. $(-6)^2$.
30. $(-1)^6$.
31. $(-1)^7$.

32. If $x = -2$, what is the value of x^2? of $-x$? of $(-x)^2$?

If $a = 4$ and $b = -3$, find the value of:

33. $a + b$.
34. $a^2 - 2b$.
35. $-b$.
36. $a^2 - \frac{1}{3}b$.
37. $(-b)^2$.
38. $3a^2b^2$.
39. $-3ab$.
40. $a^2 - b^2$.

If $a = 5$ and $b = -3$, find the value of:

41. $(a + b)(a - b)$.
42. $a^2 + 2ab + b^2$.
43. $a^3 + 3a^2 + b$.
44. $a^3 + 3a^2b + 3ab^2 + b^3$.

CHAPTER III

ALGEBRAIC EXPRESSIONS

37. Terms used in Algebra and Arithmetic. As already seen, many of the terms of mathematics are used in algebra exactly as in arithmetic. Thus we speak of addition, addends, and sum; of subtraction, minuend, subtrahend, and remainder; of multiplication and division, with the various numbers entering into these processes; of fraction, numerator, and denominator; and of various other operations and terms, these being used in algebra in the same way that they are used in arithmetic. Since they are well known, such terms usually do not require further definition, although a few are formally defined at this time for future reference.

It should always be remembered that the letters of algebra represent numbers. This is the reason why the terms used in arithmetic may properly be employed in connection with algebraic expressions.

38. Factor. Any one of two or more numbers which multiplied together form a product is called a *factor* of the product.

Thus just as 2 and 3 are factors of 6, so a and b are factors of ab; $(b + b')$ and h are factors of $(b + b')h$; m and m are factors of m^2; and 2, π, and r are factors of $2\pi r$.

39. Literal and Numerical Factors. A factor containing a letter is called a *literal* factor; one that is expressed by a numeral is called a *numerical* factor.

In most cases factors are considered to be integers. In an expression like $2\pi r$, for example, π is considered a factor because it has the form of an integer, although it equals 3.1416 – ; and r is considered a factor although its numerical value may be fractional. Similarly, we may speak of $\sqrt{5}$ as a factor of $2\sqrt{5}$, since no misunderstanding will arise from so doing. We would not, however, say that 5 is factorable, although it equals $\sqrt{5} \cdot \sqrt{5}$.

40. Coefficient. If an expression is the product of two factors, either factor is called the *coefficient* of the other.

Thus in the expression ab, a is the coefficient of b, and b is the coefficient of a.

The factor that is considered the coefficient is usually written first. Thus in $2\pi r$, 2 is the coefficient of πr, and 2π is the coefficient of r.

The coefficient 1 is omitted, x being the same as $1x$.

41. Power. The product of several equal factors is called a *power*.

Thus $2 \times 2 \times 2 = 2^3$, or 8; and 2^3, or 8, is the third power of 2. Similarly, as already stated, aaa is the third power of a, and is written a^3.

Other kinds of powers will be considered later.

42. Exponent. The number placed to the right and slightly above another to indicate a power is called an *exponent*.

Thus in a^3, 3 is the exponent of a.

In $2r^3$, 2 is the coefficient of r^3, and 3 is the exponent of r. The two should be carefully distinguished. *The coefficient shows the number of equal addends; the exponent shows the number of equal factors.*

A letter without an exponent is considered as having the exponent 1.

43. Root. One of several equal factors of an expression is called a *root*.

For example, we have square roots and cube roots, as already defined in § 7. Furthermore, since $2^4 = 16$, it follows that 2 is the fourth root of 16. That is, $\sqrt[4]{16} = 2$.

44. Absolute Term. If a polynomial contains a numerical term, this is called the *absolute term*.

Thus in the polynomial $a^2 + 3a - 4$ the absolute term is -4, and in the polynomial $3x^4 + \sqrt{2}$ the absolute term is $\sqrt{2}$.

45. Similar Terms. Monomials that have a common factor are called *similar terms* or *similar monomials*.

Thus a, $3a$, and $-7a$ are similar with respect to a; $2\sqrt{5}$ and $-\frac{2}{3}\sqrt{5}$ are similar with respect to $\sqrt{5}$; $2 \cdot (-7)$ and $4 \cdot (-7)$ are similar with respect to -7; and ax^2 and bx^2 are similar with respect to x^2.

Terms that are not similar are said to be *dissimilar*.

Exercise 29. Algebraic Expressions

Examples 1 to 31, oral — Examples 32 to 45, written

1. What are the factors of ab? of a^2? of a^2b? of $3a^2b$?

2. What are the terms of $a-b$? of $a+2b$? of $a^2-2ab+b^2$?

3. What are the numerical coefficients in a? $3a$? $-\frac{3}{4}ab^3$?

4. What are the exponents in a^2? $3a^4$? a^2b^3? $x^2y^3z^4$?

5. What other way is there of writing $a+a+a$? $a \cdot a \cdot a$?

6. What monomials form the trinomial $a^2-2ab+b^2$?

7. From $2a^2, -7a^3, -6a, -\frac{1}{2}a^2, -5a, 4a^3, 7a^2, -3a^2, 2a^3$, select four similar terms; three other similar terms.

If $a = 2$, find the value of:

8. a^2. 11. $2a$. 14. $\frac{1}{2}a$. 17. $-7a$. 20. $4a^2$. 23. $\frac{1}{2}a^3$.
9. a^3. 12. $3a$. 15. $\frac{3}{2}a$. 18. $-9a$. 21. $2a^3$. 24. $\frac{1}{4}a^4$.
10. a^4. 13. $4a$. 16. $\frac{5}{2}a$. 19. $-6a$. 22. $8a^4$. 25. $\frac{1}{8}a^5$.

If $a = 3$ and $b = -2$, find the value of:

26. $a+b$. 28. a^2-b. 30. $2a+b$.
27. $a-b$. 29. a^2+b^2. 31. $2a-b$.

If $x = 2$ and $y = -3$, find the value of:

32. $5x^2+7y^3$. 34. x^3+y^3. 36. x^4+y^4.
33. $7x^2-2y^2$. 35. $x^2+2xy+y^2$. 37. x^4+3y^4.

38. The factors of a monomial are a, x, b, a, 4, x, and y. Write the monomial in the usual way.

39. The terms of a polynomial are $3x^2y$, x^3, y^3, and $3xy^2$. Write the polynomial, beginning with the highest power of x and letting the exponents of x decrease by 1 to the right.

If $x = 10$, find the value of:

40. $2x+1$. 43. $4x^3+2x^2+3x+7$.
41. $3x^2+2x+1$. 44. $5x^3+5x^2+5x+5$.
42. $5x^2+9x+1$. 45. $x^4+x^3+x^2+x+1$.

46. Function. An algebraic expression that depends upon another for its value is called a *function* of the latter.

For example, the formula for the circumference of a circle is $c = 2\pi r$. Here the factors 2 and π are definite numbers, but r may have any value, and the value of c depends upon the value we give to r. Therefore c is called a *function* of r.

Similarly, in the area of a circle we have $a = \pi r^2$. Here a depends upon the value of r, and therefore is a function of r.

In computing interest we may use the formula $i = prt$. Here i is a function of p, a function of r, and also a function of t.

47. Symbols. If some function of r is to be referred to several times in a discussion, it is convenient to have some symbol to represent it. It is customary to use $f(r)$ as this symbol, and to read it "function of r," or simply "f of r."

Thus if we are going to speak a number of times of the expression $x^3 + 3x^2 + 3x + 1$, it is convenient to let $f(x) = x^3 + 3x^2 + 3x + 1$ and to speak of it always as $f(x)$ in the discussion. If we have two different expressions that are functions of x, we may speak of one as $f(x)$ and the other as $F(x)$, — "f minor of x" and "f major of x," respectively.

A series of dots is read "and so on," as in $x = 1, 2, 3, \cdots$, to 10.

48. Evaluation of Functions. To *evaluate* $f(x)$ is to put the given value of x in place of x in the function.

Thus if $f(x) = x^2 - x - 9$, we evaluate it for $x = 3$ by putting 3 in place of x. Then $3^2 - 3 - 9 = -3$. This is usually expressed thus:

If $\qquad f(x) = x^2 - x - 9,$

then $\qquad f(3) = 3^2 - 3 - 9 = -3.$

That is, $f(3)$ means the value of $f(x)$ when 3 is put in place of x.

Exercise 30. Evaluation of Functions

Examples 1 to 5, oral — Examples 6 to 20, written

1. If $f(x) = x + 9$, what is $f(2)$? $f(3)$? $f(-9)$?
2. If $f(x) = x^2 - 1$, what is $f(2)$? $f(4)$? $f(5)$? $f(7)$?
3. If $f(x) = x^2 + x + 1$, what is $f(1)$? $f(2)$? $f(3)$? $f(5)$?
4. If $f(r) = \frac{44}{7}r$, what is $f(7)$? $f(1)$? $f(14)$? $f(\frac{1}{7})$?
5. If $f(p) = 6\% \, p$, what is $f(100)$? $f(1000)$? $f(2000)$?

Evaluate the following functions:

6. $f(m) = 2m + 5$, for $m = 1.9$; for $m = 26$.

7. $f(x) = x^2 - x + 7$, for $x = 3$; for $x = 4$.

8. $f(y) = 2y^2 + y - 8$, for $y = 4$; for $y = 5$.

9. $f(m) = m^3 + m^2 + m + 1$, for $m = 1$; for $m = 2$.

10. $f(a) = a^3 + 3a^2 + 3a + 1$, for $a = 3$; for $a = 4$; for $a = 10$.

11. If $f(x) = x^2 + 2x + 1$, what is $f(2)$? $f(5)$? $f(7)$? $f(11)$?

12. If $f(x) = x^2 + x + 1$ and $F(x) = x^2 - x + 1$, find the value of $f(3)$ and $F(4)$, and then of $f(3) + F(4)$.

13. If $f(x) = 4x^2 + 1$, and $F(x) = 7x^2 + 2x + 3$, find the value of $f(5)$ and $F(2)$, and then of $f(5) + F(2)$.

14. If $f(x) = x^2 - 4x + 4$, which is greater, $f(2)$ or $f(1)$? $f(1)$ or $f(0)$? $f(1)$ or $f(3)$? $f(10)$ or $f(9)$?

15. If $f(x) = x^2 - 6x + 9$, which is greater, $f(0)$ or $f(2)$? $f(1)$ or $f(5)$? $f(2)$ or $f(4)$? $f(6)$ or $f(0)$?

16. If it is known that a rectangle is 10 in. long and h inches high, the area is a function of what quantity? What is the area when $h = 1$? 3? 9? $5\frac{1}{2}$? 7.4? 0?

17. If it is known that a triangle is 8 in. high and has a base b inches, the area is a function of what quantity? What is the area when $b = 5$? 7? $9\frac{1}{4}$? 5.7? 0?

18. In the formula $c = 2\pi r$, c is a function of what letter? Give the various values of this function for $r = 0, 1, 2, 3, 4, 5$, using $3\frac{1}{7}$ for the value of π in this and similar cases.

19. If $f(r) = \pi r^2$, make a table of its values for the following:

$r =$	0	1	2	3	4	5	6	7	8	10
$f(r) =$										

20. Make a similar table showing the values of $f(x) = x^2 + 1$, for $x = 0, 1, 2, \cdots, 10$.

Exercise 31. Evaluating Expressions

Examples 1 to 23, oral — Examples 24 to 48, written

1. What is the value of $2+2+2$? of $3\cdot 2$? of $2\cdot 2\cdot 2$? of 2^3?

2. What is the value of $8 \div 4 + 4$? of $8 \div (4+4)$?

Simplify the following:

3. $-7-2$.
4. $-7-9$.
5. $-9-7$.
6. $16 \div 2 + 2$.
7. $16 \div (2+2)$.
8. $16 \div (2 \times 2)$.
9. $(2+3)(3+2)$.
10. $(5+2)(5-2)$.
11. $(2+3)(3+4)$.

If $a=2$ and $b=4$, find the value of:

12. $\frac{a}{b}$.
13. $\frac{b}{a}$.
14. $\frac{a^2}{b}$.
15. $\frac{a+b}{b}$.
16. $\frac{b-a}{b}$.
17. $\frac{b+a}{b-a}$.
18. $\frac{a^2+b}{8}$.
19. $\frac{a+b^2}{9}$.
20. $\frac{a^3-b}{4}$.
21. $\frac{2a}{b}$.
22. $\frac{3a}{a+b}$.
23. $\frac{a^3-b}{8}$.

If $a=5$ and $b=2$, find the value of:

24. $0.2a$.
25. $0.5b$.
26. $0.7ab$.
27. $0.1ab^2$.
28. $0.2ba^2$.
29. $0.3a^3b$.
30. $6a^2+b^2$.
31. $6a^2-b^2$.
32. $25a^3+b^3$.
33. $7a^b$.
34. $7b^a$.
35. a^{b+1}.

If $a=6$, $b=-4$, and $c=2$, find the value of:

36. $\frac{a+b}{c}$.
37. $\frac{a+c}{b}$.
38. $\frac{b+c}{a}$.
39. $\frac{a+b+c}{a}$.
40. $\frac{a+b+c}{b}$.
41. $\frac{a+b+c}{c}$.
42. $\frac{a^2+b^2}{c^2}$.
43. $\frac{a^2-b^2}{c^2}$.
44. $\frac{a^2+2ab+b^2}{c^4}$.

If $x=2$, $y=3$, and $z=-4$, find the value of:

45. $x+y+z$.
46. $3x+2y+5z$.
47. $3x^2-2y^2-5z^2$.
48. $2xy+3yz+4zx$.

Exercise 32. Representing Expressions

Examples 1 to 5, oral — Examples 6 to 16, written

1. If a represents the greater of two numbers and b the less, how will you represent their sum? their difference?

2. Using the same letters as in Ex. 1, how will you represent the sum of the squares of two numbers? the difference of their squares? the product of their cubes?

3. How will you represent twice some number a? 1 more than twice the number? 1 less than twice the number?

4. How will you represent the square of the sum of two numbers? the square of the difference of two numbers?

5. Using r for radius and c for circumference, how will you state the product of the radius and circumference? half of this product?

Letting a, b, and c represent the three numbers, express algebraically the following:

6. The sum of three numbers.

7. The sum of the squares of three numbers.

8. Five times the product of three numbers.

9. The square of the first of three numbers divided by the sum of the other two.

10. The sum of the first two numbers divided by three times the cube of the third.

If the three dimensions of a rectangular box are l, w, and h, express algebraically the following:

11. The volume of the box.

12. The area of each side and of the bottom.

13. The sum of the four edges that run lengthwise.

14. The sum of the four edges that run crosswise.

15. The sum of the four vertical edges.

16. The sum of all twelve edges of the box.

Exercise 33. Evaluating Expressions

Examples 1 to 5, oral — Examples 6 to 14, written

1. If $a = 7$ and $b = 9$, find the value of a^2; of $a^2 - b$; of $a^2 + b$.

2. If $m = 5$ and $n = 30$, find the value of $m \div n$; of $n \div m$; of $36 m \div 9$.

3. If $x = 8$ and $y = -4$, find the value of $2y$; of $2y \div x$; of $y^2 \div x$; of $x - 2y$.

4. How do you represent 5 times one number plus 7 times the same number?

5. How do you represent 5 times one number, plus 7 times another number, plus 3 times another number?

Letting x, y, and z represent the three numbers, express algebraically the following:

6. The sum of three numbers. Evaluate for $x = y = z = 7$.

7. The sum of the squares of three numbers. Evaluate for $x = 1$, $y = 2$, $z = -3$; for $x = 0.1$, $y = 0.2$, $z = -0.3$.

8. The sum of the cubes of three numbers. Evaluate for $x = 1$, $y = 3$, $z = 5$; for $x = 7$, $y = 9$, $z = 8$.

9. The product of three numbers. Evaluate for $x = 4$, $y = -3$, $z = 10$; for $x = 17$, $y = -9$, $z = 24$.

10. The product of the squares of three numbers. Evaluate for $x = 5$, $y = 6$, $z = -7$; for $x = 0.1$, $y = 0.6$, $z = -0.9$.

11. The sum of three numbers divided by their product. Evaluate for $x = 4$, $y = -6$, $z = 10$; for $x = 7$, $y = 9$, $z = -3$.

12. The sum of the first two numbers diminished by the third number. Evaluate for $x = -10$, $y = 35$, $z = 17$.

13. The first number diminished by the sum of the second and third numbers. Evaluate for $x = 72$, $y = 29$, $z = -16$.

14. If $f(x) = x^2 + 1$, and $F(x) = x^2 - 2x + 15$, find the value of $f(2)$ and $F(5)$. Then find the value of $f(2) \cdot F(5)$; of $F(5) \div f(2)$.

CHAPTER IV

ADDITION

Exercise 34. Addition of Similar Monomials

Examples 1 to 22, oral — Examples 23 to 31, written

1. Add 2 ft. and 3 ft.; $2f$ and $3f$; $2 \cdot 5$ and $3 \cdot 5$.

2. Add 7¢ and 9¢; $7c$ and $9c$; $7 \cdot 2$ and $9 \cdot 2$; $7 \cdot 10$ and $9 \quad 10$

3. Add $15 and $11; $15d$ and $11d$; $15 \cdot 2$ and $11 \cdot 2$.

4. Add 3 in., 4 in., and 8 in.; $3i$, $4i$, and $8i$.

5. 7 mi. + 14 mi.

6. $7m + 14m$.

7. $7 \cdot 2 + 14 \cdot 2$.

8. 6 bu. + 4 bu. + 9 bu.

9. $6b + 4b + 9b$.

10. $6 \cdot 7 + 4 \cdot 7 + 9 \cdot 7$.

11. How do you proceed to add similar monomials?

12. $1 + $17.

13. $d + 17d$.

14. $x^2 + 17x^2$.

15. $ab + 15ab$.

16. $x^2y^2 + 27x^2y^2$.

17. $abc + 32abc$.

18. Add -7, -3, and 11; $-7x$, $-3x$, and $11x$.

19. $-x + 2x$.

20. $-y + 13y$.

21. $-x - 2x - x$.

22. $-7x - 4x - x$.

23. $-2ab + 3ab + 4ab - 7ab + 10ab$.

24. $-21xyz + 30xyz + 2xyz - 3xyz$.

25. $2\sqrt{2} + 3\sqrt{2} + 4\sqrt{2} + 5\sqrt{2} + 10\sqrt{2} + 3\sqrt{2}$.

26. $4p + 5p$.

27. $4(a + b) + 5(a + b)$.

28. $5(a - b) + 2(a - b)$.

29. $4(a + b) - 2(a + b)$.

30. $-3(a - b) + 7(a - b)$.

31. $-2\sqrt{a + b} + 3\sqrt{a + b}$.

49. Algebraic Sum. The result obtained by adding two or more numbers considered with respect to their signs as well as their values, is called their *algebraic sum.*

Thus the algebraic sum of 2 and -3 is -1, although $2+3=5$.

50. Addition of Monomials. *To add similar monomials, find the algebraic sum of the coefficients of the common factor and prefix this sum to the common factor.*

For this purpose we may consider an expression like $27(a-b)$ as a monomial, as in Ex. 18, below. In case a letter has no coefficient, 1 is understood. Thus $5x+x=6x$.

To add dissimilar monomials, write the terms one after the other, each with its proper sign.

Thus the sum of a, $2b$, and $-c$ is $a+2b-c$.

Exercise 35. Addition of Monomials

Examples 1 to 11, oral — Examples 12 to 19, written

1. Add 2 ft. and 3 ft.; $2f$ and $3f$; $2 \cdot 25$ and $3 \cdot 25$.

2. Add 3 yd., 2 ft., and 7 in., expressing the result as a compound number. Add $3y$, $2f$, and $7i$; $3a$, $2b$, and $7c$.

3. Add a, $-2b$, and $3c$; x, $-2y$, and $3z$; m, $-2n$, and $3p$.

4. Add $-2a$, b, and $4c$; $-2x$, y, and $4z$; $-3x$, y, and z.

5. Add $5a$, $6b$, and $-7c$; $5p$, $6q$, and $-7r$; $7a$, $4b$, and c.

Add the following:

6. $a, -a.$
7. $-a, a.$
8. $19a, -3x, 3x.$
9. $-4a, 42x, 4a.$
10. $17a, -19x, 19x.$
11. $-17a, -19x, 17a.$
12. $18a^2b, 36a^2b, -7a^2b.$
13. $2\sqrt{a}, 13\sqrt{a}, 15\sqrt{a}.$
14. $16\sqrt{m}, -23\sqrt{m}, \sqrt{m}.$
15. $92abc, 37abc, -75abc.$
16. $15mn^2, -7mn^2, 16mn^2.$
17. $36pqr, 49pqr, -17pqr.$
18. $27(a-b), 38(a-b), 33(a-b), -17(a-b).$
19. $24(x^2+y^2), 17(x^2+y^2), -15(x^2+y^2), 46(x^2+y^2).$

Exercise 36. Addition of Polynomials

Examples 1 to 9, oral — Examples 10 to 17, written

1. Add 2 ft. 3 in. and 3 in.; $2f + 3i$ and $3i$.

2. Add 7 yd. 2 ft. and 6 yd.; $7y + 2f$ and $6y$.

Add the following:

3.

$$\begin{array}{l} 6 \text{ ft.} + 7 \text{ in.} \\ \underline{2 \text{ ft.} + 3 \text{ in.}} \end{array}$$

4.

$$\begin{array}{l} 6f + 7i \\ \underline{2f + 3i} \end{array}$$

5.

$$\begin{array}{r} 6 \cdot 5 + \ \ 7 \cdot 8 \\ \underline{2 \cdot 5 + \ \ 3 \cdot 8} \\ (?) \cdot 5 + (?) \cdot 8 \end{array}$$

6.

$$\begin{array}{l} 2 \text{ rd.} + 8 \text{ ft.} + 9 \text{ in.} \\ \underline{4 \text{ rd.} + 3 \text{ ft.} - 2 \text{ in.}} \end{array}$$

7.

$$\begin{array}{l} 2r + 8f + 9i \\ \underline{4r + 3f - 2i} \end{array}$$

8.

$$\begin{array}{l} 2 \cdot 3 + 8 \cdot 7 + 9 \cdot 4 \\ \underline{4 \cdot 3 + 3 \cdot 7 - 2 \cdot 4} \end{array}$$

9.

$$\begin{array}{l} x^2 + 10xy + 15z^2 \\ \underline{x^2 + \ \ 4xy - 10z^2} \end{array}$$

10.

$$\begin{array}{l} 3w + 2x + 3y + 4z \\ \underline{2w - 3x + 2y + 3z} \end{array}$$

11.

$$\begin{array}{l} x^2 + 10y + 7z + 9 \\ \underline{x^2 - \ \ 3y + 4z - 9} \end{array}$$

12.

$$\begin{array}{r} 7^2 + \ \ 5 \cdot 7 \cdot 3 + \ \ \ \ 3^2 \\ \underline{7^2 - \ \ 2 \cdot 7 \cdot 3 + \ \ \ \ 3^2} \\ (?) \cdot 7^2 + (?) \cdot 7 \cdot 3 + (?) \cdot 3^2 \end{array}$$

13.

$$\begin{array}{l} x^4 + x^2y^2 + y^4 + x^2 + y^2 \\ \underline{x^4 - x^2y^2 + y^4 - x^2 - y^2} \end{array}$$

14.

$$\begin{array}{l} a + 3b - c + d - e + f \\ \underline{a - 3b + c - d + e - f} \end{array}$$

15.

$$\begin{array}{l} a^3 - 3a^2b + 3ab^2 - b^3 \\ \underline{a^3 + 3a^2b + 3ab^2 + b^3} \end{array}$$

16.

$$\begin{array}{r} x^3 + 3x^2y + 3xy^2 + y^3 \\ \underline{-x^3 + 3x^2y - 3xy^2 + y^3} \end{array}$$

17. If $f(x) = x^3 + 2x^2 - 7x + 1$ and $F(x) = 2x^3 + x^2 + 10x + 2$, what does $f(x) + F(x)$ equal?

51. Addition of Polynomials. It is evident from the preceding exercise that

To add polynomials we write similar terms in the same column and add these terms, writing their sums as a polynomial.

52. Check. An operation that tends to prove the correctness of another operation is called a *check* upon that operation.

Thus in addition we check by adding in the other direction.

53. Checks in Algebra. One of the best checks on the operations in algebra is the substitution of any values we please for the letters, as in the following example, where we let $x=1$, $y=1$, and $z=1$.

OPERATION	CHECK
$2x+3y-4z+6$	$2+3-4+6=7$
$3x-7y+8z-3$	$3-7+8-3=1$
$5x-4y+4z+3$	$5-4+4+3=8$

Here we have simply put 1 in place of x, y, and z, in the addends and in the sum, and we have $7+1=8$. Therefore the work *checks*.

We may have an error in spite of this check, as would be the case if we should write $5y-4x+4z+3$ instead of $5x-4y+4z+3$, or if we should make an error in computation in the check. In case of doubt, especially in case of exponents, use other values than 1.

Exercise 37. Addition of Polynomials

Examples 1 to 3, oral — Examples 4 to 28, written

1. Add $2x+y$ and $3x+y$; $2x+y$ and $3x-y$.
2. Add $7x+3y$ and $3x+7y$; $7x+3y$ and $-3x-7y$.
3. Add $a+b+c$ and $a-b+c$; $a-b-c$ and $a+b+c$.

Add the following and check the results:

4. x^2+xy+y^2, $x^2-2xy+y^2$, $x^2+2xy+y^2$.
5. x^2+y^2, x^2-y^2, $2x^2+y^2$, x^2+2y^2, x^2, $3y^2$.
6. $x^3+x^2y+xy^2+y^3$, $2x^3-3x^2y+7xy^2-6y^3$.

Add the following and check the results:

7. $5x^2 + 3x + 7,\ 6x^2 - 3x + 7,\ x + 2.$

8. $7x^2 - 9x + 2,\ 6x^2 - 8x + 3,\ x - 9.$

9. $2x^2 + 3x - 9,\ 4x^2 - 6x - 8,\ x + 4.$

10. $3x^2 + 5x + 6,\ 7x^2 - 7x + 9,\ x - 4.$

11. $4x^2 - 9x - 8,\ 2x^2 - x + 1,\ x^2 + 3x - 1.$

12. $-x^2 + 2x + 7,\ -x^2 - x + 9,\ 3x^2 + 2x + 3.$

13. $-5x^2 - 7x + 9,\ -10x^2 - 14x + 13,\ 2x^2 - 8.$

14. $a^2 + ab + b^2,\ a^2 - ab + b^2,\ -a^2 + ab - b^2,\ a^2 + b^2.$

15. $a^3 + a^2 + a + 1,\ a^3 - a^2 + a - 1,\ a^3 + a^2 - a + 1,\ -3a^3.$

16. $p^2 + q^2 + 4,\ p^2 - q^2 + 7,\ -p^2 + q^2 - 8,\ -p^2 - q^2 - 3.$

17. $m^2n + mn^2 + 7,\ m^2n + mn^2 - 7,\ m^2n - 2mn^2,\ -3m^2n.$

Simplify the following by combining like terms:

18. $a^2 + 3b^2 - 4c^2 + 2d^2 - 7 + 3a^2 - 4b^2 + c^2 - 2d^2 + 7.$

19. $a^3 + 3a^2b + 3ab^2 + b^3 - a^3 + 3a^2b - 3ab^2 - b^3$

20. $x^3 + 2x^2 + 3x - 1 + 4x + 3x^2 + 1 - x^3 - 5x^2 - 7x.$

21. $m^2n + mn^2 + mn - 3mn^2 - mn + 4m^2n.$

22. Add $3t^3 + 5t^2 + 4t + 2$ and $2t^3 + 3t^2 + 2t + 5$; also 3542 and 2325. What do the polynomials equal if $t = 10$?

23. How much is 2 ft. + 3 ft.? a ft. + b ft.? $2f + 3f$? $af + bf$? $ax + bx$? $am + bm$? $2 \cdot 5$ and $3 \cdot 5$? $a \cdot 5$ and $b \cdot 5$?

We may think of a ft. and b ft. as $(a + b)$ft. Similarly, $af + bf = (a + b)f$. That is, we do not know the numerical value of the coefficients a and b, so we *indicate* their sum.

Add the following, indicating the sums of the coefficients:

24. p yd. + q yd.; $py + qy$; $pm + qm$; $p \cdot 5 + q \cdot 5$.

25. $am + bm$; $ax + bx$; $axy + bxy$; $a \cdot 2 \cdot 3 + b \cdot 2 \cdot 3$.

26. $m\sqrt{a} + n\sqrt{a}$; $ma^2 + na^2$; $m\sqrt{a+b} + n\sqrt{a+b}$.

27. $ax + bx + cx$; $am^2 + bm^2 + cm^2$; $a\sqrt{m} + b\sqrt{m} + c\sqrt{m}$.

28. $x\sqrt{2} + y\sqrt{2} - z\sqrt{2}$; $x\sqrt{a-b} + y\sqrt{a-b} - z\sqrt{a-b}$.

54. Equations involving Addition. We have already studied equations involving addition, subtraction, multiplication, and division. We shall use all of these processes as necessary, but shall now consider some special features in addition.

Required to solve the equation $x + 3x + 4x = 64$.

Since $x + 3x + 4x = 64,$
therefore $8x = 64,$ by uniting terms,
and $x = 8,$ by dividing equals by 8.

Check. Substituting 8 in the *original equation*, we have

$$8 + 3 \cdot 8 + 4 \cdot 8 = 8 + 24 + 32 = 64.$$

Exercise 38. Equations involving Addition

Examples 1 to 8, oral — Examples 9 to 13, written

1. Solve the equation $x + 2x = 9$. How will you check?

2. Solve the equation $x + 4x = 25$. Check the result.

Solve the following and check the results:

3. $x + 7x = 72.$
4. $2x + 5x = 49.$
5. $3x + 2x = 75.$
6. $x + 2x + 3x = 6.$
7. $2x + x + 4x = 14.$
8. $3x + 3x + 3x = 27.$

9. A rectangle is twice as long as wide. If w represents the width, represent the length in terms of w. Represent the perimeter in terms of w. Draw the figure.

10. A rectangle is twice as long as wide. The perimeter is 12 in. What is the width? the length?

11. A rectangle is three times as long as wide. The perimeter is 24 in. What is the width? the length?

12. A rectangular field is four times as long as wide. The perimeter is 100 rd. What is the width? the length?

13. A rectangular box is twice as wide as deep, and twice as long as wide. If the sum of the twelve edges is 28 in., what is the depth? the width? the length?

CHAPTER V

SUBTRACTION

Exercise 39. Subtraction of Similar Monomials

Examples 1 to 17, oral — Examples 18 to 26, written

1. How much is 7 ft. − 4 ft.? $7f - 4f$? $7 \cdot 2 - 4 \cdot 2$?

2. How much is 10 mi. − 3 mi.? $10m - 3m$? $10a - 3a$?

3. How much is 16 yd. − 9 yd.? $16y - 9y$? $16 \cdot 3 - 9 \cdot 3$?

4. How much is 27 rd. − 14 rd.? $27r - 14r$? $27 \cdot 5 - 14 \cdot 5$?

State the value of:

5. \$30 − \$12.

6. $30d - 12d$.

7. $30a - 12a$.

8. $15x - 7x$.

9. $15ax - 7ax$.

10. $15\sqrt{a} - 7\sqrt{a}$.

11. $29a - 11a$.

12. $29x^2 - 11x^2$.

13. $29\sqrt{xy} - 11\sqrt{xy}$.

14. How do you subtract one monomial from a similar monomial?

15. How much is a ft. − b ft.? a mi. − b mi.? $ax - bx$?

16. From 7° subtract 6°; from 7° subtract 7°; from 7° subtract 8°. What must be added to 6 to make 7? to 7 to make 7? to 8 to make 7?

17. How many degrees from −1° to +7°? How much is $7° - (-1°)$? $7 - (-1)$? $7a - (-a)$?

State the value of:

18. $4.7a - 3.9a$.

19. $5.1x^2 - 2.8x^2$.

20. $7.2mn - 3.9mn$.

21. $3x - 2\frac{4}{5}x$.

22. $13\frac{3}{4}x - 5\frac{1}{2}x$.

23. $12\frac{1}{2}y - 7\frac{7}{8}y$.

24. $7° - (-3°)$.

25. $7d - (-3d)$.

26. $7x - (-3x)$.

55. Subtraction of Monomials. It is evident from the preceding exercise that

To subtract a monomial from a similar monomial, we find the difference between the coefficients of the common factor and multiply this difference by the common factor.

Thus, $9a - 4a = 5a$, $ax - bx = (a - b)x$, and $mx^2 - 4nx^2 = (m - 4n)x^2$.

In case a letter has no coefficient expressed, 1 is understood as usual. Thus $5x - x = 4x$.

If the monomials are dissimilar we indicate the subtraction.

Thus, as we may write 7 ft. − 3 in., so we write $7f - 3i$, or $7x - 3y$, or $7\sqrt{a+b} - 3\sqrt{x+y}$.

Exercise 40. Subtraction of Monomials

Examples 1 to 3, oral — Examples 4 to 16, written

1. What must be added to 3 to make 7? to $3a$ to make $7a$? to $3a^2b$ to make $7a^2b$? How much is $7a^2b - 3a^2b$?

2. What must be added to $-3°$ to make $7°$? to $-3x$ to make $7x$? to $-3\sqrt{a}$ to make $7\sqrt{a}$? How much is $7a - (-3a)$?

3. What must be added to $-10°$ to make $4°$? to $-10x$ to make $4x$? How much is $4x - (-10x)$?

State the value of:

4. $30\sqrt{a} - 17\sqrt{a}$.
5. $5.1x^2 - 3.9x^2$.
6. $7\frac{1}{2}m - 3\frac{2}{3}m$.
7. $7\frac{1}{3}m - 5\frac{1}{2}m$.
8. $8\frac{2}{3}q - 7\frac{3}{4}q$.
9. $9a - 9\frac{3}{4}a$.
10. $7.1ab - 3.8ab$.
11. $9.3a^2x - 5.7a^2x$.
12. $a^2b^2 - ya^2b^2$.

13. From the sum of $4a$ and $7a$ subtract $-5a$.

14. From the sum of $5xy$ and $9xy$ subtract $-12xy$.

15. From the sum of $7ab$ and $8ab$ subtract the sum of $-2ab$, $15ab$, and $-9ab$.

16. From the sum of $6x^2$, $9x^2$, and $-3x^2$, subtract the sum of $17x^2$, $-6x^2$, $-9x^2$, and $3x^2$.

56. Subtraction as the Inverse of Addition. We may think of subtraction as the operation by which we find the number that added to the subtrahend will equal the minuend.

Thus, because 3 added to 4 makes 7, therefore $7 - 4 = 3$.
Because 7 added to -3 makes 4, therefore $4 - (-3) = 7$.
Because 5° added to -9° makes -4°, therefore $-4^\circ - (-9^\circ) = 5^\circ$.

We therefore have the following cases in subtraction:

$$4 + 5 = 9, \text{ therefore } 9 - 5 = 4;$$
$$4 + (-5) = -1, \text{ therefore } -1 - (-5) = 4;$$
$$-4 + 5 = 1, \text{ therefore } 1 - 5 = -4;$$
$$-4 + (-5) = -9, \text{ therefore } -9 - (-5) = -4.$$

Therefore, *to subtract one quantity from another, find a quantity that added to the subtrahend will equal the minuend.*

If the student will think of this when he subtracts, he will rarely be troubled in the matter of signs. If he is confused, he should, as already stated, remember that in taking a smaller number from a larger one the result must be positive, and in taking a larger number from a smaller one the result must be negative.

Exercise 41. Subtracting Monomials

Examples 1 to 16, oral — Examples 17 to 26, written

1. $3 - 4$.
2. $3 - (-4)$.
3. $-3 - 4$.
4. $-3 - (-4)$.
5. $3a - 4a$.
6. $3a - (-4a)$.
7. $-3a - 4a$.
8. $-3a - (-4a)$.
9. $5ax - 3ax$.
10. $5ax - (-3ax)$.
11. $7m^2 - 4m^2$.
12. $7m^2 - (-4m^2)$.
13. $-6c - 7c$.
14. $-6c - (-7c)$.
15. $-9x - 10x$.
16. $-9x - (-10x)$.
17. $73a - 19a$.
18. $37a - (-19a)$.
19. $-71a^2 - 39a^2$.
20. $-71a^2 - (-39a^2)$.
21. $51ab - 27ab$.
22. $62ab - (-35ab)$.
23. $ax^2 - (-bx^2)$.
24. $a^2x^2y^2 - (-b^2x^2y^2)$.

25. From the sum of $3x$ and $7x$ subtract $5x - 2x$.

26. From the difference between $37.5\,ax$ and $29.3\,ax$ subtract the difference between $42.9\,ax$ and $40.7\,ax$.

Exercise 42. Subtracting Polynomials

Examples 1 to 11, oral — Examples 12 to 25, written

1. From 4 ft. 8 in. subtract 2 ft. 3 in.

2. From 8 ft. – 4 in. subtract 2 in.; subtract 6 ft. 2 in.

3. If the thermometer registers + 2° and the temperature falls 4°, what does it then register?

4. If the thermometer registers – 2° and the temperature falls 4°, what does it then register?

Subtract:

5.
7 lb. 4 oz.
3 lb. 2 oz.

6.
$7x + 4y$
$3x + 2y$

7.
$7 \cdot 3 + 4 \cdot 2$
$3 \cdot 3 + 2 \cdot 2$

8.
$9m + 48f$
$4m + 20f$

9.
$9a^2b + 48c$
$4a^2b + 20c$

10.
15 ft. – 4 in.
8 ft. + 2 in.

11.
$15x - 4y$
$8x + 2y$

12.
$8a + 4b$
$3a + 2b$

13.
$8a + 4b$
$5a + 4b$

14.
$8 \cdot 9 + 4 \cdot 5$
$5 \cdot 9 + 4 \cdot 5$

15.
$15m + 7n$
$9m + 6n$

16.
$15p + 7q$
$12p + 7q$

17.
$15p + 7q$
$14p + 8q$

18.
$20x - 8y$
$13x + 9y$

19.
$17x^2y^2 + 15xy$
$12x^2y^2 + 14xy$

20.
$23a^2b - 15b^2c$
$17a^2b + 16b^2c$

21.
$23 \cdot 10 - 15 \cdot 3$
$17 \cdot 10 + 16 \cdot 3$

22.
$71ax + mby$
$47ax + nby$

23.
$29x^2y + axy^2$
$30x^2y + bxy^2$

24.
$xm^2n - 21mn^2$
$ym^2n + 15mn^2$

25.
$ax^2y^2 + cxy$
$bx^2y^2 + dxy$

57. Subtracting Polynomials. From the preceding exercise we see that

To subtract one polynomial from another, we arrange similar terms under one another and subtract these terms separately.

For example, subtract $4a^2-3ab+1$ from $6a^2-8-9ab$. Rearranging, we have the following:

$$\begin{array}{ccc} \text{Operation} & \qquad & \text{Check} \\ 6a^2-9ab-8 & & 6-9-8=-11 \\ \underline{4a^2-3ab+1} & & \underline{4-3+1}=\underline{\quad 2} \\ 2a^2-6ab-9 & & 2-6-9=-13 \end{array}$$

Here the work is checked by letting $a=1$ and $b=1$. If a check upon the exponents is desired, use other values than 1 for a and b.

Exercise 43. Subtracting Polynomials

Examples 1 to 6, oral — Examples 7 to 13, written

1. From $4a+3b$ take $2a+3b$.
2. Subtract $6m-3n$ from $7m-3n$.
3. From $9x-7y+1$ take $10x-8y+1$.

Subtract and check:

4.
$$\begin{array}{r} 12ab-15cd \\ \underline{11ab-15cd} \end{array}$$

5.
$$\begin{array}{r} 16ax-17ay \\ \underline{14ax-18ay} \end{array}$$

6.
$$\begin{array}{r} 19mn-15pq \\ \underline{14mn-25pq} \end{array}$$

7.
$$\begin{array}{r} x^2+xy+y^2 \\ \underline{x^2-xy+y^2} \end{array}$$

8.
$$\begin{array}{r} a^2-2ab+b^2 \\ \underline{a^2+2ab-b^2} \end{array}$$

9.
$$\begin{array}{r} 2x^2-4xy+7y^2 \\ \underline{2x^2+4xy-7y^2} \end{array}$$

10.
$$\begin{array}{r} a^3+3a^2+4a \\ \underline{a^3-3a^2-4a} \end{array}$$

11.
$$\begin{array}{r} 4a^2+3a-4 \\ \underline{2a^2-4a+7} \end{array}$$

12.
$$\begin{array}{r} 6x^2+5x-7 \\ \underline{4x^2-7x-9} \end{array}$$

13. From $4a^3+7a^2b-6ab^2-8b^3$ take $3a^3-9ab^2+7b^3-6a^2b$.

Exercise 44. Subtracting Polynomials

Examples 1 to 8, oral — Examples 9 to 32, written

1. From a take $-a$; from $a+b$ take $a-b$.

2. From $-b$ take b; from $a-b$ take $a+b$.

Subtract:

3. a^2-b^2 from a^2+b^2.

4. a^2+b^2 from a^2-b^2.

5. $-a^2-b^2$ from a^2+b^2.

6. $ax-1$ from $ax+1$.

7. $ax+1$ from $ax-1$.

8. $1-ax$ from $ax-1$.

9. From $7x-2y+3$ take $4x+2y-3$.

10. From $9a+3b-4c$ take $8a+4b-7c$.

11. From $6a^2-9b^2+7c^2$ take $5a^2-8b^2-6c^2$.

12. From $17ab-19cd+25ef$ take $20ab-30cd+40ef$.

Rearrange the terms properly and subtract:

13. $a^2+c^2-b^2$ from $a^2+b^2-c^2$.

14. $a^2-3c^2+7b^2$ from $6b^2-7c^2-9a^2$.

15. $ax+cz-by$ from $ax+2by-3cz$.

16. $a^2b^2+m^2n^2-x^2y^2$ from $3m^2n^2+4x^2y^2-5a^2b^2$.

17. $a^2+b^2-c^2+4d^2$ from $3a^2-7d^2-9c^2$.

18. From $3a$ ft. $+ 2b$ in. take a ft. $+ b$ in.

19. From $3af$ take af; from $3af+2bi$ take $af+bi$.

20. From $xf+yi$ take $af+bi$; from $xf+y$ take $wf+zy$.

If $A=2a^2+3ab-b^2$, $C=-a^2+5ab$,
$B=a^2-3ab+b^2$, $D=-5a^2-7ab-9b^2$,

find the expression for:

21. $A-B$.

22. $C-D$.

23. $A-C$.

24. $A-D$.

25. $A+B-C$.

26. $A+C-B$.

27. $B+C-A$.

28. $B-A-C$.

29. $A+B+C+D$.

30. $A+B-C+D$.

31. $A-B+C-D$.

32. $A-B-C-D$.

Exercise 45. Equations involving Subtraction

Examples 1 to 26, oral — Examples 27 to 53, written

1. How much is $4x - 3x$? $4x - x$? $4x - 4x$? $4x - 5x$?

2. If $4x - 3x = 7$, what does x equal? Prove it.

3. If $7x - x = 30 + 6$, what does x equal? Prove it.

Solve these equations:

4. $6x - 2x = 24$.
5. $9x - 3x = 42$.
6. $7x - 2x = 35$.
7. $8x - 3x = 30$.
8. $9a - a = 72$.
9. $7a - a = 72$.
10. $6a - a = 75$.
11. $5a - a = 64$.
12. $12y - 7y = 5$.
13. $15y - 8y = 49$.
14. $17y - 9y = 72$.
15. $36y - 7y = 29$.

16. In the equation $7x + 3 = 24$ what must first be subtracted from both members? By what do we then divide?

Solve these equations:

17. $7x + 4 = 39$.
18. $7x + 9 = 79$.
19. $9x + 5 = 50$.
20. $4x + 3 = 43$.
21. $5x + 7 = 22$.
22. $6x + 7 = 19$.
23. $8x + 3 = 27$.
24. $8x + 9 = 33$.
25. $8x + 7 = 87$.

26. How do you prove that your result is correct in the solution of an equation? Illustrate.

Solve these equations and check the results:

27. $9x - x = 16$.
28. $2x + \frac{3}{4} = 2\frac{3}{4}$.
29. $7x - x = 24$.
30. $7x - 2x = 15$.
31. $8x - 5x = 45$.
32. $8x - 7x = 50$.
33. $6x - 3x = 75$.
34. $6x - 3x = 78$.
35. $5x - 3x = 50$.
36. $6x + \frac{3}{4} = 6\frac{1}{4}$.
37. $6x + 9 = 64$.
38. $7x + 8 = 57$.
39. $7x + 8 = 58$.
40. $8x + 7 = 71$.
41. $8x + 7 = 75$.
42. $9x + 6 = 60$.
43. $9x + 6 = 63$.
44. $9x + 6 = 66$.
45. $15a - 3a = 60$.
46. $15a - 3a = 66$.
47. $17a - 4a = 26$.
48. $17a - 4a = 27$.
49. $21y - 7y = 28$.
50. $21y - 7y = 35$.
51. $33m - 4m = 29$.
52. $33m - 4m = 32$.
53. $32p - 7p = 80$.

Exercise 46. Equations involving Subtraction

Examples 1 to 14, oral — Examples 15 to 27, written

1. In the equation $6x = 10 + x$, what must first be subtracted from both members? By what do we then divide?

Solve these equations:

2. $2x = 9 + x$.

3. $3x = 8 + x$.

4. $4x = 9 + x$.

5. $5x = 8 + x$.

6. $7x = 10 + 2x$.

7. $8x = 12 + 2x$.

8. $9x = 36 + 3x$.

9. $9x = 40 + 4x$.

10. $12x = 81 + 3x$.

11. $13x = 4.5 + 4x$.

12. $14x = 2.1 + 7x$.

13. $15x = 3.5 + 8x$.

14. In the equation $7x + 5 = 4x + 17$, what literal term should we first subtract from both members? Then what numerical term? By what do we then divide?

Solve these equations:

15. $9x + 1 = 2x + 15$.

16. $9x + 3 = 3x + 27$.

17. $9x + 5 = 5x + 25$.

18. $1.2x + 4 = 0.9x + 16$.

19. $1.4x + 3 = 0.8x + 39$.

20. $1.5x + 0.8 = 0.9x + 6.8$.

21. If I add 3 to 12 times a certain number, the result is 123. What is the number?

22. If I add 5 to 8 times a certain number, the result is 29. What is the number?

23. If I add 7 to 6 times a certain number, the result is 43. What is the number?

24. If from 6 times a certain number I take twice the number, the result is 48. What is the number?

25. If from 10 times a certain number I take 7 times the number, the result is 39. What is the number?

26. If 7 times your age plus 4 times your age is 165 years, how old are you?

27. If from 9 times a certain number I take 7 times the number, the result is 96. What is the number?

58. Problems in Percentage. Problems in percentage are often more easily solved by algebra than by arithmetic. For example:

1. If 15% of a number is 9165, what is the number?

Solution by Arithmetic

Since 15% of the number $= 9165$,

therefore 1% of the number $= \frac{1}{15}$ of 9165, or 611,

and the number $= 100 \times 611$, or 61,100.

Solution by Algebra

Let x represent the number.

Then $0.15x = 9165$,

and $x = 61{,}100$, by dividing both members by 0.15.

2. What number increased by $66\frac{2}{3}\%$ of itself equals 275?

Solution by Arithmetic

Since 100% of the number = the number,

therefore $66\frac{2}{3}\%$ of the number = the increase,

and $166\frac{2}{3}\%$, or $\frac{5}{3}$, of the number $= 275$.

Therefore $\frac{1}{3}$ of the number $= \frac{1}{5}$ of 275, or 55,

and the number $= 3 \times 55$, or 165.

Solution by Algebra

Let x represent the number.

Then $x + 0.66\frac{2}{3}x = 275$,

and $x = 165$, by dividing both members by $1.66\frac{2}{3}$.

3. After deducting 10% from the marked price of some goods, a dealer sold them for \$13.50. What was the marked price?

Let x represent the *number* of dollars of marked price.

Then $x - 0.10x = 13.50$,

or $0.90x = 13.50$.

Therefore by dividing equals by 0.90, $x = 13.50 \div 0.90$

$= 15$.

Therefore the marked price was \$15.

Check. \$15 − 10% of \$15 = \$15 − \$1.50 = \$13.50.

Exercise 47. Percentage

Examples 1 to 10, oral — Examples 11 to 24, written

1. A number less 10% of itself is what per cent of itself?

2. A number plus 10% of itself is what per cent of itself?

State the per cent of x in the following:

3. $x + 0.05x$. **5.** $x + 20\% x$. **7.** $x + 0.50x$. **9.** $x + 75\% x$.
4. $x - 0.05x$. **6.** $x - 20\% x$. **8.** $x - 0.50x$. **10.** $x - 75\% x$.

11. What number less 10% of itself equals 72?

12. What number less 17% of itself equals 166?

13. $435 is 6% of what sum of money?

14. $19.75 is 5% of what sum of money?

15. $38.25 is 4½% of what sum of money?

16. What is the number of which 14.4 is 66⅔%?

17. What is the sum of which $41.25 is 15%?

18. What is the sum of which 33⅓% is $2.25?

19. A certain number increased by 12½% of itself equals 819. What is the number?

20. A boy now weighs 84 lb., which is 12% more than he weighed a year ago. How much did he weigh then?

21. A certain school gained 15% this year over the number last year. It now has 161 pupils. How many had it last year?

22. A dealer saved $1968 this year from his store. This is 18% less than he saved last year. How much did he save last year?

23. A dealer was obliged to sell some damaged furniture at 10% less than cost. He sold it for $85.50. How much did it cost? How much did he lose?

24. A farmer sold his milk to a factory where he received credit for 1045 lb. of butter fat. If his milk tested 3.8% butter fat, how many pounds of milk did he sell?

59. Removal of Parentheses preceded by the Positive Sign. If to \$4 we add \$3 + \$2, we have in all \$4 + \$3 + \$2, or \$9. It is evidently of no consequence whether we add \$2 to the sum of \$4 and \$3, or add \$5 to \$4.

Hence $\$4 + (\$3 + \$2) = \$4 + \$3 + \$2$.

Similarly, $5¢ + (2¢ + 7¢) = 5¢ + 2¢ + 7¢$,

$$a + (b + c) = a + b + c,$$

and $$a + (b - c) = a + b - c.$$

If an expression inclosed within parentheses is preceded by the sign +, the parentheses may be removed without any change in the signs of the terms.

Similarly, we may have any other sign of aggregation.

Thus $7 + \overline{3 + 4} = 7 + 3 + 4$, and $12 + [5 - 2] = 12 + 5 - 2 = 15$.

Exercise 48. Removal of Parentheses

Examples 1 to 10, oral — Examples 11 to 18, written

1. How much is \$4 plus the sum of \$3 and \$9? How much is the sum of \$4, \$3, and \$9?

Remove the parentheses and simplify the results:

2. $7 + (9 + 6)$.
3. $7 + (9 - 6)$.
4. $8 + (15 + 5)$.
5. $8 + (15 - 5)$.
6. $4a + (3a + a)$.
7. $5a + (7a + 3a)$.
8. $7a + (7a - 3a)$.
9. $6x^2y^2 + (9x^2y^2 - x^2y^2)$.

10. How much is the sum of 7 and 3 increased by the sum of 4 and 2? How much is the sum of 7, 3, 4, and 2?

Remove the parentheses and simplify the results:

11. $(2a + 3a) + (7a + a)$.
12. $(2a + 3a) + (7a - a)$.
13. $(2a - 3a) + (7a + a)$.
14. $(2a - 3a) + (7a - a)$.
15. $(6x + 9y) + (5x - 3y)$.
16. $a - b + (a + b)$.
17. $a^2 - b^2 + (a^2 + b^2)$.
18. $5a + 17 + (5a - 17)$.

60. Removal of Parentheses preceded by the Negative Sign. If we subtract $b + c$ from a, and $b - c$ from a, we have the following results:

$$\begin{array}{l} a \\ \underline{b + c} \\ a - b - c \end{array} \qquad\qquad \begin{array}{l} a \\ \underline{b - c} \\ a - b + c \end{array}$$

We therefore see that

$$a - (b + c) = a - b - c,$$

and

$$a - (b - c) = a - b + c.$$

If an expression inclosed within parentheses is preceded by the negative sign, the parentheses may be removed provided the sign before each term is changed.

That is, $x + y - (x - y) = x + y - x + y = 2y$;

and $$\frac{3}{2} - \frac{5-4}{2} = \frac{3-5+4}{2} = \frac{2}{2} = 1,$$

the fraction bar having the force of parentheses.

If $f(x) = x^2 + 2x - 1$,

and $F(x) = x^2 - 7x - 8$,

then $f(x) - F(x) = x^2 + 2x - 1 - x^2 + 7x + 8 = 9x + 7$,

and $F(x) - f(x) = x^2 - 7x - 8 - x^2 - 2x + 1 = - 9x - 7$.

Exercise 49. Removal of Parentheses

Examples 1 to 8, oral — Examples 9 to 48, written

1. Subtract $7 - 5$ from 10. Subtract 7 from 10 and add 5.
2. Subtract $7a - 5a$ from $10a$. How much is $10a - 7a + 5a$?

Remove the parentheses and simplify the results:

3. $20 - (8 + 2)$.
4. $20 - (8 - 2)$.
5. $30 - (9 - 4)$.
6. $4a - (3a - a)$.
7. $7x - (4x - 2x)$.
8. $9x - (5x - 3x)$.

9. If $f(x) = x^2 - 3x - 15$, and $F(x) = x^2 - 9x - 11$, what is the value of $f(x) - F(x)$? of $F(x) - f(x)$?

10. In Ex. 9 what is the value of $x^2 - f(x)$? of $x^2 + 7 - f(x)$?

11. In Ex. 9 what is the value of $f(x) - (x^2 - 5x + 7) - F(x)$?

12. In Ex. 9 what is the value of $F(x) - (x^2 + 9x - 6) - f(x)$?

13. How much is $(17a + 15b) - (12a - b)$?

Remove the parentheses and simplify the results:

14. $12x - (9x + 7x)$.

15. $17y - (12y - 4y)$.

16. $39a - (15a + b)$.

17. $47a - (15b + 7a)$.

18. $56n + (27n - 4n)$.

19. $39ab + (ab + 1)$.

20. $56a^2 - (3a^2 + 7)$.

21. $48p + (27p - 2)$.

22. $63q - (49 + 4q)$.

23. $79x - (79 - 79x)$.

24. How much is $25a^2 - \overline{2a^2 + 1}$? $25a^2 + \overline{2a^2 + 1}$?

Remove the bars and simplify the results:

25. $4a - \overline{3a - a}$.

26. $7a - \overline{4a + 2a}$.

27. $8a + \overline{6a - 3a}$.

28. $9x - \overline{7x - 2x}$.

29. $ab + \overline{3 - ab}$.

30. $xy - \overline{7 + xy}$.

31. $pq - \overline{5 - pq}$.

32. $x^2y^2 - \overline{x^2y^2 - 1}$.

Remove the brackets and simplify the results:

33. $27a^2 - [4 - 27a^2]$.

34. $35ab - [5 + 35ab]$.

35. $42xy - [-xy + 42]$.

36. $75x^2y^2 + [-x^2y^2 + 7]$.

37. $80mn + [75 - 80mn]$.

38. $[35x + 1] - 1$.

39. $[42 - x] + x$.

40. $-[27 + ab] - ab$.

41. $-[xy - 24] + 24$.

42. $-[mn + 75] + 75$.

Remove the parentheses and brackets and simplify the results:

43. $[a + b] - (a - b)$.

44. $[a - b] - (a + b)$.

45. $[a + b] + (-a - b)$.

46. $-[a + b] - (-a - b)$.

47. $-[a - b] - (-a + b)$.

48. $[x^2 + y^2 - z^2] - (x^2 - y^2 + z^2)$.

61. Removal of Several Symbols of Aggregation. The symbols of aggregation most frequently used in algebra are the following:

Parentheses, as in $a - (b + c)$; *Brackets*, as in $p - [q - r]$;
Bar or *vinculum*, as in $x - \overline{y - z}$; *Braces*, as in $m - \{m + n\}$.

When one symbol of aggregation incloses another, we may remove either the outer one or the inner one first. Beginners usually find it less confusing to remove the inner one first.

1. Required to simplify the expression $10 - (4 - \overline{3 - 2})$.

$$\begin{aligned} 10 - (4 - \overline{3 - 2}) &= 10 - (4 - 3 + 2) \\ &= 10 - 4 + 3 - 2 \\ &= 7. \end{aligned}$$

2. Required to simplify the expression $20a - [10a - (a - b)]$.

$$\begin{aligned} 20a - [10a - (a - b)] &= 20a - [10a - a + b] \\ &= 20a - [9a + b] \\ &= 20a - 9a - b \\ &= 11a - b. \end{aligned}$$

3. Required to simplify the expression

$$20a - \{10a - [6a - \overline{c - (5a - b)}] + c\}.$$

$$\begin{aligned} &20a - \{10a - [6a - \overline{c - (5a - b)}] + c\} \\ &= 20a - \{10a - [6a - \overline{c - 5a + b}] + c\} \\ &= 20a - \{10a - [6a - c + 5a - b] + c\} \\ &= 20a - \{10a - 6a + c - 5a + b + c\} \\ &= 20a - 10a + 6a - c + 5a - b - c \\ &= 21a - b - 2c. \end{aligned}$$

Problems like Ex. 3 are so rarely found in algebra that they may be omitted unless the student is preparing for some examination in which they are required.

62. Insertion of Parentheses. From what we have learned of the removal of parentheses we see that

Two or more terms may be inclosed in parentheses preceded by a plus sign without changing the signs of the terms.

Two or more terms may be inclosed in parentheses preceded by a minus sign, provided the sign of each of the terms is changed.

Exercise 50. Removal of Symbols of Aggregation

Examples 1 to 4, oral — Examples 5 to 22, written

1. Simplify $a - (b + c) + (b - c)$.

2. Simplify $a - (b - c) - (b + c)$.

3. Simplify $b - (c - d)$; $a - [b - (c - d)]$.

4. How do you remove several symbols of aggregation?

Remove the symbols of aggregation and simplify:

5. $12a - (7a - b) + (6a - 4b)$.

6. $17xy - (15xy + 4) - (2xy - 8)$.

7. $25ab - (5ab + c) + (5ab - c)$.

8. $(a - b) - (b - c) - (c - d) - (d - e) - (e - a)$.

9. $a - (b - c) + b - (c - d) + c - (d - a)$.

10. $a^2 + 2ab + b^2 - (a^2 - 2ab + b^2)$.

11. $a^2 - (2ab - b^2) - [a^2 - (2ab + b^2)]$.

12. $x^2 - [x^2 - (x^2 - 2xy + y^2) + 2xy - y^2]$.

13. $a^3 - (3a^2b - 3ab^2) + b^3 - [a^3 - (3a^2b - 3ab^2 + b^3)]$.

14. $a + 7 - (2a - 7) - [a - 7 - (2a - \overline{3a - 7})]$.

15. $x^3 - (3x^2y - 3xy^2) + [y^3 - x^3 + (3x^2y - 3xy^2 + y^3)]$.

16. $a - [a + b - (c - d + e - a) + c] - b - c + d - e$.

17. $2a - \{3a + b - c - 4c + [3a - (b - \overline{c - 2b})]\}$.

18. $7a - \{2a - b + c + 4c - [4a + (a - b + 3c)]\}$.

Remove the parentheses, leaving the brackets:

19. $[a^2 - (b^2 + c^2)] \times [a^2 - (b^2 - c^2)]$.

20. $[ab - (cd + 1)] \times [ab - (cd - 1)]$.

21. In the expression $a^2 - 2ab + b^2$ inclose the last two terms in parentheses, arranging the signs so as not to change the value of the expression.

22. In the expression $ax + bx + cx - px^2 - qx^2 - rx^2$ inclose the last three terms in parentheses without changing the value.

Exercise 51. Equations involving Parentheses

Examples 1 to 5, oral — Examples 6 to 24, written

1. Solve the equation $6x = 49 - x$.

2. Solve the equation $5x = 49 - 2x$.

3. Solve the equation $5x = 50 - (1 + 2x)$.

4. Solve the equation $4x = 60 - (11 + 3x)$.

5. Solve the equation $10x = 64 - (4 + 2x)$.

Solve the following equations:

6. $10x = 28 - (5x - 2)$.

7. $14x = 36 - (3 - 3x)$.

8. $16x = 85 - (4x + 5)$.

9. $25x = 39 - (5x + 9)$.

10. $27x = 80 - (5 - 2x)$.

11. $30x = 60 - (3x - 6)$.

12. $30x = 99 - (x + 6)$.

13. $17x = 85 - (3x + 5)$.

14. $31x = 80 - (5 - 6x)$.

15. $31x = 67 - (1 + 2x)$.

16. If from 25 I subtract a certain number diminished by 5, the result is 14 times the number. Required the number.

17. If from 30 I subtract a certain number diminished by 3, the result is 10 times the number. Required the number.

Solve the following equations:

18. $25x - 12 = 8x - (-7 + 2x)$.

19. $34x - 11 + x = 9x - (-3 - x) + x$.

20. $75x - 15 - x = 60x - (5 - 10x) - x$.

21. $90x + 10 + 2x = 60x - (-50 - 10x) + 2x$.

22. Solve the equation $27x - (3x - 9) = 3x + 30$.

23. If to twice a certain number I add 7, and take this sum from 60, I have 7 less than 10 times the number. Required the number.

24. If to three times a certain number I add 19, and take this sum from 93, the result is one fourth of the sum of twice the number and 16. Required the number.

CHAPTER VI

MULTIPLICATION

Exercise 52. Multiplication of Monomials

Examples 1 to 20, oral — Examples 21 to 38, written

1. Multiply by 5: 2 mi.; 2 ft.; $2m$; $2f$; $2 \cdot 3$; $2x$.

2. Multiply by 7: 3 yd.; $3y$; 3 times a given number; $3n$.

3. Multiply by 9: 4 in.; $4i$; \$4; $4d$; 4¢; $4c$; $4x^2y^2$; $4 \cdot 7$.

4. If the temperature is 3° below zero, what will it be when it is twice as much below zero?

5. How much is $2 \times (-3°)$? $2 \times (-4°)$? $2 \times (-3d)$? $2 \times (-4d)$? $2 \times (-4 \text{ lb.})$? $2 \times (-4l)$?

6. $2 \times 12a$.

7. $2 \times 12ab$.

8. $2 \times (-12)$.

9. $2 \times (-12a^2)$.

10. $3 \times 20x$.

11. $3 \times 20xy$.

12. $3 \times (-20)$.

13. $3 \times (-20x^2)$.

14. $4 \times 50m$.

15. $4 \times 50m^3$.

16. $4 \times (-50)$.

17. $4 \times (-50\sqrt{a})$.

18. How do you multiply a monomial by a positive integer?

19. How may $a \times a$ be written more briefly? $a \times a \times a$? $a \times a^2$? $a^2 \times a$? $a \times a^3$? $a^2 \times a^2$? $a^3 \times a^2$?

20. State a rule for multiplying a^3 by a^4.

21. $3 \times 4a^2$.

22. $a \times 4a^2$.

23. $3a \times 4a^2$.

24. $5 \times 6x^3$.

25. $x^2 \times 6x^3$.

26. $5x^2 \times 6x^3$.

27. $7 \times 17m^2$.

28. $7m \times 17m^2$.

29. $8 \times 36x^3$.

30. $8x \times 36x^3$.

31. $9x^2 \times 36x^2$.

32. $9x^2 \times 48x^3$.

33. $15 \times 17a$.

34. $15a \times 17a$.

35. $15a^3 \times 17a^5$.

36. $27a^3 \times 17a^6$.

37. $35a^4 \times 27a^5$.

38. $56y^4 \times 92y^4$.

63. Laws of Exponents. Since a^2 means $a \cdot a$, and a^3 means $a \cdot a \cdot a$, we see that

$$a^2 \cdot a^3 = a \cdot a \times a \cdot a \cdot a = a^{2+3} = a^5.$$

Furthermore, $$a^m \cdot a^n = a^{m+n}.$$

For $a^m = a$ taken m times as a factor,

and $a^n = a$ taken n times as a factor.

Therefore $a^m \cdot a^n = a$ taken $m + n$ times as a factor.

In multiplying monomials, the exponent of any letter in the product is equal to the sum of the exponents of that letter in the factors.

Since $(a^2)^3$ means $a^2a^2a^2$, or a^6 and $(a^m)^n$ means $a^ma^ma^m \cdot$ (n times), therefore

$$(a^m)^n = a^{mn}.$$

That is, $(a^3)^5 = a^{15}$, $(x^7)^3 = x^{21}$, and so on.

64. Law of Signs. The law of signs in multiplication is given on page 40. Briefly stated it is as follows:

In multiplication, two like signs produce plus, two unlike signs produce minus.

That is, $+a \cdot (+b) = +ab$, $+a \cdot (-b) = -ab$,

$-a \cdot (-b) = +ab$, $-a \cdot (+b) = -ab$.

Therefore $$2x^2y \cdot 5x^3y^2 = 10x^5y^3,$$

$$2xy \cdot (-5x^2y^2) = -10x^3y^3.$$

Evidently, therefore, *the product of an even number of negative factors is positive, and the product of an odd number of negative factors is negative.*

65. Degree of a Term. The number of literal factors in a term is called the *degree of the term.*

Thus a^2 is of the second degree, a^3 is of the third degree, a^m is of the mth degree, and a^{m+3} is of the $m + 3$ degree. Similarly, x^2y^3 is of the fifth degree, because $xxyyy$ has five literal factors.

We may, however, speak of x^2y^3 as of the second degree in x, or as of the third degree in y, if we mention specifically the letter.

66. Degree of a Polynomial. The degree of the term that is of the highest degree in a polynomial is called the *degree of the polynomial.*

Thus $a^2x^2 + bx + c$ is of the fourth degree, because a^2x^2 is of the fourth degree. It is, however, of the second degree in x, because a^2x^2 is of the second degree in x.

Similarly, $a^4x^2 + 1$ is of the sixth degree, but it is of the second degree in x and of the fourth degree in a.

67. Homogeneous Polynomial. A polynomial of which all of the terms are of the same degree is said to be *homogeneous.*

Thus $x^3 + 3x^2y + 3xy^2 + y^3$ is homogeneous, because every term is of the third degree.

Similarly, $x^4 + x^2y^2 + y^4$ is a homogeneous trinomial of the fourth degree.

68. Multiplication of Monomials. As we have seen in the exercise on page 73, to find the product of two monomials we may proceed as follows:

Find the product of the numerical coefficients, writing after this product the letters, each letter having an exponent equal to the sum of its exponents in the factors.

It will be found better to write the product in this order: the sign of the product; the product of the numerical coefficients; the letters in their alphabetical order, each with its proper exponent.

For example,

$$4ab \cdot 5ab = 4 \cdot 5 \cdot a \cdot a \cdot b \cdot b = 20a^2b^2;$$

$$3a^mb \cdot 4a^nb = 3 \cdot 4 \cdot a^m \cdot a^n \cdot b \cdot b = 12a^{m+n}b^2;$$

$$(-7x^7y^6)^2 = -7 \cdot (-7) \cdot x^7 \cdot x^7 \cdot y^6 \cdot y^6 = 49x^{14}y^{12};$$

$$-6abc \cdot (-7a^2b^2c^2) = 6 \cdot 7 \cdot a \cdot a^2 \cdot b \cdot b^2 \cdot c \cdot c^2 = 42a^3b^3c^3;$$

$$-5p^qr^s \cdot (-4p^2r^3) = 5 \cdot 4 \cdot p^q \cdot p^2 \cdot r^s \cdot r^3 = 20p^{q+2}r^{s+3}.$$

In practice the result should be written down rapidly, or stated orally, without the intermediate step here given.

In multiplying a by b, instead of saying "*a times b*" it is common to say "*a into b.*" The expression is a very old one and the word "into," used in this sense, has lost its original meaning, but the student will often hear it used in algebra.

Exercise 53. Multiplication of Monomials

Examples 1 to 45, oral — Examples 46 to 62, written

1. a^2a^9.
2. b^8b^8.
3. c^9c^4.
4. d^7d^7.
5. e^9e^8.
6. m^8m^{11}.
7. p^4p^{12}.
8. $a^{11}a$.
9. $a^{12}a$.
10. a^ma.
11. a^ma^4.
12. a^ma^p.
13. x^ax^b.
14. a^xa^y.
15. $a^{2x}a^x$.
16. $a^{3x}a^x$.
17. $a^{3x}a^{2x}$.
18. $ab \cdot ab$.
19. $a^2b \cdot a^2b$.
20. $a^2b^8 \cdot a^2b^3$.
21. $a^4b^7 \cdot a^5b^2$.
22. $p^9q^7 \cdot p^7q^9$.
23. $5\,a^7 \cdot 6\,a^5$.
24. $7\,a^8 \cdot 8\,a^7$.
25. $9\,x^m \cdot 7\,x$.
26. $9\,x^m \cdot 8\,x^2$.
27. $8\,x^m \cdot 5\,x^n$.
28. $-\,a^2 \cdot 3\,a^4$.
29. $-\,a^8 \cdot 5\,a^7$.
30. $-\,2\,a^4 \cdot 6\,a^6$.
31. $3\,a \cdot (-\,4\,a)$.
32. $3\,a \cdot (-\,4\,a^2)$.
33. $3\,a^m \cdot (-\,5\,a^n)$.
34. $ab \cdot (-\,ab)$.
35. $-\,ab \cdot ab$.
36. $-\,ab \cdot (-\,ab)$.
37. $x^2y^2 \cdot (-\,2\,xy)$.
38. $-\,3\,xy^3 \cdot (-\,7\,xy)$.
39. $-\,4\,x^2y^3 \cdot (-\,5\,x^2y)$.
40. $-\,6\,x^2y^5 \cdot 10\,x^3y^3$.
41. $abc \cdot abc$.
42. $a^2b^2c^2 \cdot abc$.
43. $a^mbc \cdot a^nbc$.
44. $(a^2b^2c^2d^2)^3$.
45. $(p^3q^3r^3s^3)^2$.

46. If the circumference of a circle is πd, what is the sum of the circumferences of 7 circles of the diameter d?

47. If the circumference of a circle, πd, is multiplied by d, what is the result? What is the result if it is multiplied by $\frac{1}{2}\,d$?

48. If an edge of a cube is $4\,x$, what is the volume of the cube?

49. $-\,27\,x^m \cdot (-\,23\,x^7)$.
50. $-\,42\,x^{17} \cdot (-\,34\,x^9)$.
51. $-\,68\,p^m \cdot (-\,27\,p^n)$.
52. $-\,47\,p^x \cdot (-\,31\,p^y)$.
53. $-\,96\,p^x \cdot (-\,41\,p^{2x})$.
54. $(17\,x^2)^3$; $(96\,x^3)^3$.
55. $(24\,x^4)^2$; $(73\,x^7)^2$.
56. $37\,x^{4m} \cdot 67\,x^{9m}$.
57. $83\,x^{m+1} \cdot 75\,x^{m-1}$.
58. $43\,a^{p+4} \cdot 72\,a^{p-4}$.
59. $-\,49\,x^{a+b} \cdot 32\,x^{a-b}$.
60. $-\,26\,x^{a+3b} \cdot 34\,x^{a-5b}$.
61. $72\,a^{2x+1} \cdot 33\,a^{4x-2}$.
62. $34\,x^{a-7} \cdot (-\,42\,x^{4a+9})$.

69. Multiplication of a Polynomial by a Monomial. If we multiply 5 ft. 2 in. by 3, we have 15 ft. 6 in. In the same way, if we multiply 5 times one number and 2 times another number by 3, we have 15 times the first plus 6 times the second. That is,

$$\begin{array}{lll}
5\text{ ft. }2\text{ in.} & 5f+2i & 5x+2y \\
3 & 3 & 3 \\
\hline
15\text{ ft. }6\text{ in.} & 15f+6i & 15x+6y
\end{array}$$

In the same way we have the following:

OPERATION	CHECK
$a^2 \quad -2ab \quad +3b^2$	$1-2+3=2$
$\underline{ab\qquad\qquad\qquad}$	$\underline{1\qquad\quad}=\underline{1}$
$a^3b-2a^2b^2+3ab^3$	$1-2+3=2$

It should be noticed that in algebra it is more convenient to write the multiplier at the left and work from left to right.

In this check we let $a=1$ and $b=1$. This checks the coefficients, where the error is most liable to occur, but it does not check the exponents, since any power of 1 is 1. If a check upon the exponents is desired, let $a=2$ and $b=2$, or take other values. In case either factor becomes zero in the check, use some other values for the letters.

It is also a good check to notice that if both multiplicand and multiplier are homogeneous, the product will also be homogeneous.

To multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial and add the partial products.

Exercise 54. Multiplying by a Monomial

Examples 1 to 6, oral — Examples 7 to 12, written

1. $a(b+c)$.
2. $-a(b-c)$.
3. $3a^2(a+b)$.
4. $-3a^2(a-b)$.
5. $a(b+c-d)$.
6. $-a(b-c+d)$
7. $2x^2(x^2-2xy+y^2)$.
8. $4x^3(x^2-3x+27)$.
9. $-7ab(a^2-2ab+b^2)$.
10. $-9a^2b(a^2-2ab+b^2)$.
11. $35a^3x^5(a^7-3a^4x^2+2ax^5)$.
12. $-8x^ny^n(x^ny^n-7xy-13)$.

70. Multiplication of a Polynomial by a Polynomial. If we multiply 43 by 21, we multiply first by 1 unit and then by 2 tens, and then add the partial products, thus:

Multiplicand	43	$40+3$
Multiplier	21	$20+1$
Multiplying by 1 unit	43	$40+3$
Multiplying by 2 tens	86	$800+\ 60$
Sum of partial products	903	$800+100+3$

In a similar manner we multiply $3a^2-b$ by a^2+b, thus:

	Operation	Check
	$3a^2-b$	$3-1=2$
	a^2+b	$1+1=2$
Multiplying by a^2	$3a^4-a^2b$	4
Multiplying by b	$3a^2b-b^2$	
	$3a^4+2a^2b-b^2$	$3+2-1=4$

To multiply a polynomial by a polynomial, multiply the multiplicand by each term of the multiplier, and add the partial products.

Exercise 55. Multiplying by a Monomial or Binomial

Examples 1 to 10, oral — Examples 11 to 20, written

1. $a(a+b)$.
2. $b(a+b)$.
3. $-4a(a-b)$.
4. $-7a^2(a-b)$.
5. $-9a^2(a^2-b^2)$.
6. $15(2xy+1)$.
7. $15(2xy-2)$.
8. $-8(4x^2y-3)$.
9. $-7(7x^2y^2-9)$.
10. $-12x(5x^m-3x^n)$.
11. $(a+b)(a-b)$.
12. $(a+b)(a+b)$.
13. $(a^2-b^2)(a-b)$.
14. $(a^2-b^2)(a^2+b^2)$.
15. $(a-7)(a^2-7a+1)$.
16. $(2x-1)(4x^2+x-1)$.
17. $(3x-2)(3x^2-4x+7)$.
18. $(a^2b^2+1)(a^2b^2+4)$.
19. $(abc-1)(a^2b^2c^2+5)$.
20. $(a^3b^2c-2)(a^3b^2c+2)$.

71. Arranging a Polynomial. A polynomial in which the exponents of a certain letter in the successive terms decrease from left to right is said to be *arranged according to the descending powers* of that letter.

Thus $x^3 + 3x^2 - 4x + 1$ is arranged according to the descending powers of x, and $an^3 + a^2n - a^3$ is arranged according to the descending powers of n.

Similarly, a polynomial may be arranged according to the *ascending* powers of a letter. Thus $a^3 - 3a^2b + 3ab^2 - b^3$ is arranged according to the ascending powers of b.

In multiplying it simplifies the work if the polynomials are arranged according to the ascending or descending powers of some letter.

Multiply $4a^3 - 2a^2 + 7 - a$ by $3 + a^2 - 3a$.

OPERATION CHECK

Rearranging, we multiply thus:

$$\begin{array}{rrrrrrr}
4a^3 & -\ 2a^2 & -\ a & +\ 7 & & & =8 \\
a^2 & -\ 3a & +\ 3 & & & & =1 \\
\hline
4a^5 & -\ 2a^4 & -\ a^3 & +\ 7a^2 & & & 8 \\
 & -\ 12a^4 & +\ 6a^3 & +\ 3a^2 & -\ 21a & & \\
 & & 12a^3 & -\ 6a^2 & -\ 3a & +\ 21 & \\
\hline
4a^5 & -\ 14a^4 & +\ 17a^3 & +\ 4a^2 & -\ 24a & +\ 21 & =8
\end{array}$$

In the operation we multiply first by a^2, then by $-3a$, and finally by 3. We then add the partial products.

In the check we have let $a = 1$. We have then only to add the coefficients, having $4 - 2 - 1 + 7 = 8$, and $1 - 3 + 3 = 1$, and similarly for the product.

Multiply $x^2 - 3xy + 3y^2$ by $y^2 + x^2$.

Rearranging, we multiply thus:

$$\begin{array}{rrrrrr}
x^2 & -\ 3xy & +\ 3y^2 & & & =1 \\
x^2 & +\ y^2 & & & & =2 \\
\hline
x^4 & -\ 3x^3y & +\ 3x^2y^2 & & & 2 \\
 & & x^2y^2 & -\ 3xy^3 & +\ 3y^4 & \\
\hline
x^4 & -\ 3x^3y & +\ 4x^2y^2 & -\ 3xy^3 & +\ 3y^4 & =2
\end{array}$$

Exercise 56. Multiplying by a Polynomial

Examples 1 to 4; oral — Examples 5 to 50, written

1. Arrange $a^2 + a^3 + 1 + a$ for multiplying.

2. Arrange $x + 1 + x^4 - x^2 + 3x^5 + x^6 + x^8$ according to ascending powers of x.

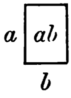
3. Arrange $x^2 + y^2 + 2xy$ according to descending powers of x. How is it then arranged with respect to y?

4. Apply the check of $x = 1$, $y = 1$, and find if $x^2 + 3xy + y^2$ can be the product of $x + y$ and $x + y$.

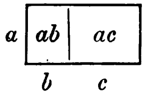
Multiply and check:

5. $(p + q)(p + q)$.
6. $(x + y)(x + y)$.
7. $(m + n)(m + n)$.
8. $(a + b)(a + 2b)$.
9. $(a + 2b)(a - 3b)$.
10. $(x^2 + 1)(x^2 + 1)$.
11. $(p + q)(p - q)$.
12. $(x + y)(x - y)$.
13. $(x^2 + 1)(x^2 - 1)$.
14. $(x^2 + y^2)(x^2 + 2y^2)$.
15. $(xy + 3)(xy + 4)$.
16. $(xyz + 1)(xyz - 5)$.

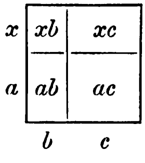
17. Since a rectangle with length b and width a has an area ab, show that these squares and rectangles illustrate the products indicated beneath:



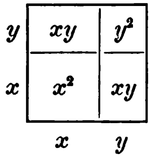
$a \times b = ab$



$a(b + c)$
$= ab + ac$



$(x + a)(b + c)$
$= xb + ab + xc + ac$



$(x + y)(x + y)$
$= (?)$

18. Multiply the product of $a + b$ and $a - b$ by $a^2 + b^2$.

19. Multiply the product of $x + y$ and $x + y$ by $x^2 - y^2$.

20. Multiply the product of $m + 7$ and $m - 7$ by $m^2 + 49$.

21. Multiply the product of $2 + y$ and $2 - y$ by $4 + y^2$.

Multiply and check:

22. $a^2 + 2ab + b^2$ by $a + b$.

23. $a^2 - 2ab + b^2$ by $a - b$.

24. $x^2 + 2xy + y^2$ by $x^2 + 2xy + y^2$.

25. $x^3 + 3x^2y + 3xy^2 + y^3$ by $x + y$.

26. $x^2 + 2xy + y^2$ by $x^2 - 2xy + y^2$.

27. $x^3 - x^2y + xy^2 - y^3$ by $x + y$.

28. $x^3 + x^2y + xy^2 + y^3$ by $x - y$.

29. $2x^3 + 3x^2y - 4xy^2 + 3y^3$ by $x^2 - 2xy + 4y^2$.

30. $4m^3 + 3m^2n - 7mn^2 + n^3$ by $m^2 + n^2$.

31. $6m^4 + 3m^2 + 2m + 1$ by $m^2 + 3m + 2$.

32. $7p^3 + 8p^2 + 6p - 4$ by $p^2 - 4p + 1$.

33. $8x^3 - 4x^2 + 7x - 6$ by $x^2 - 5x - 3$.

34. $a^2b^2 + 4ab - 7$ by $a^2b^2 - 7ab + 9$.

35. $a^2x^3 - ax^2 + x - 7$ by $a^2x^3 - x + 4$.

36. $a^2 + x^2 - ax + 5$ by $a^2 - x^2 + ax - 5$.

37. $a^3 + b^3 - a^2b + ab^2$ by $a^2 - ab + b^2$.

38. $5a^5 + 4a^4 + 3a^3 + 2a + 1$ by $2a + 1$.

39. $3x^2y - x^3 + y^3 - 4xy^2$ by $x^2 + y^2 + 4xy$.

40. $5m^3 + 7m^2 - 9m + 1$ by $m^2 - 4 + m$.

41. $x^3 - 3x^2 + 7 - 4x$ by $x^3 + 3 - 5x + 2x^2$.

42. $a^2b^2c^2 + a^3b^3c^3 - abc + 7$ by $a^2b^2c^2 + 1$.

43. $a^4x^4 + 1 - a^3x^3 - a^2x^2 - ax$ by $a^3x^3 - ax + 3$.

44. $25a^2 + 50ab + 1$ by $25a^2 - 50ab + 1$.

45. $a^2b^2x^2 - 4abx + 1$ by $a^2b^2x^2 + 4abx + 1$.

46. $a^n + a^{n-1} + a^{n-2}$ by a; also by a^2; also by $a + 1$.

47. $a^m + b^m$ by $a^m - b^m$, and this product by $a^{2m} + b^{2m}$.

48. $x^m - x^{m-1} + x^{m-2} - x^{m-3}$ by $x + 1$.

49. $x^{m+1} + x^m + x^{m-1} + x^{m-2}$ by $x + 1$.

50. $a^{2m} - 2a^mb^m + b^{2m}$ by $a^{2m} + 2a^mb^m + b^{2m}$.

Exercise 57. Equations involving Multiplication

Examples 1 to 3, oral — Examples 4 to 27, written

1. If $x^2 + 2 = x^2 + x$, what should be subtracted from both members to obtain the value of x? What does x equal?

2. If $x^2 + 2 = x(x+1)$, what is the first step in the solution? the next step? What does x equal?

3. If $(x+1)(x+2) = x(x+4)$, what is the first step in the solution? the next step? How will you check the result?

Solve the following equations, checking the result by substituting it in the original equation:

4. $x^2 + 6 = x(x+3)$.
5. $x^2 + 15 = x(x+5)$.
6. $x^2 + 28 = x(x+7)$.
7. $x^2 - 36 = x(x-4)$.
8. $x^2 - 36 = x(x-6)$.
9. $x^2 - 36 = x(x-2)$.
10. $3x(x+1) = 3x^2 + 6$.
11. $5x(x+2) = 5x^2 + 20$.
12. $7x(x+4) = 7x^2 + 28$.
13. $6x(x-3) = 6x^2 - 36$.
14. $4x(x-7) = 4x^2 - 56$.
15. $x(2x-4) = 2x^2 + 40$.

16. If $(n^2 + 2n + 1)(n^2 - 2n + 1) = n^4 - 2n^2 + n$, what is the value of n? Check by substituting in the original equation.

Solve the following equations, checking as before:

17. $(n+1)(n+2) = n(n+2) + 4$.
18. $(a+2)(a+3) = a(a+4) + 7$.
19. $(p+3)(p-3) = p^2 + p + 12$.
20. $(q+4)(q+5) = 2(4q+5) + q^2$.
21. $(x^2 + 2x + 1)(x+3) = x^2(x+5) + 6(x+1)$.
22. $(x^2 - x - 1)(2x+4) = 2x^2(x+1) - 7(x-1)$.
23. $(x^2 - 3x - 7)(x-1) = x^2(x-4) - 5(x-2)$.
24. $(x^2 - 2x - 6)(x+1) = x^2(x-1) - 9(x-1)$.
25. $(x^2 + 4x + 4)(x+1) = x^2(x+5) + 7(x+2)$.
26. $(x^2 - x + 1)(x+1) + x = (x+2)(x^2 - 2x + 4)$.
27. $(x^2 + x + 1)(x-1) + 3x = (x+2)(x^2 - 2x + 4)$.

Exercise 58. Review

Examples 1 to 6, oral — Examples 7 to 64, written

1. Add $2x+1$ and $2x-1$; also $ax+1$ and $ax-1$.
2. Add $3x+4$ and $5x+6$; also $ax+b$ and $cx+d$.
3. Subtract a^2-b^2 from a^2+b^2; also a^2-b^2 from a^2.
4. Subtract $a^2+2ab+b^2$ from $a^2-2ab+b^2$.
5. Multiply $a+b$ by a; by b; by ab; by a^2.
6. Multiply $2a-b$ by a; by $-a$; by b; by $-b$.

Given $f(x)=x^2-7x+9$, *and* $F(x)=x-7$, *find:*

7. $f(x)+F(x)$.
8. $F(x)-f(x)$.
9. $F(x)\cdot f(x)$.
10. $f(x)-5\cdot F(x)$.
11. $F(x)-7\cdot f(x)$.
12. $6\cdot F(x)-5\cdot f(x)$.
13. $2\cdot f(x)\cdot F(x)$.
14. $3\cdot F(x)\cdot F(x)$.
15. $F(x)\cdot F(x)\cdot f(x)$.
16. $F(x)\cdot f(x)\cdot f(x)$.

Given $f(x)=x^3+4x^2-5x-9$, *and* $F(x)=x^2-x-6$, *find:*

17. $f(x)+F(x)$.
18. $f(x)-F(x)$.
19. $F(x)-f(x)$.
20. $(x-2)\cdot F(x)$.
21. $(x-4)\cdot f(x)$.
22. $(x+12)\cdot f(x)$.
23. $(x^2-x)\cdot F(x)$.
24. $(x^3+1)\cdot F(x)$.
25. $3x\cdot f(x)$.
26. $(5x-1)\cdot F(x)$.
27. $(7x-9)\cdot f(x)$.
28. $f(x)\cdot F(x)$.

Simplify:

29. $(a+b)(a-3b)-(2a+b)(3a-b)$.
30. $(a+b+c)(a+b-c)-(a^2+b^2-c^2)$.
31. $(a+b+c)(a+b-c)(a-b+c)$.
32. $(a-b)(b-c)+(b-c)(c-d)+(c-d)(d-a)$.
33. $(x+1)(x+2)+(x+2)(x+3)-(2x+3)(x-7)$.
34. $(x+y)(y+z)-(z+w)(x+w)-(x+z)(y-w)$.
35. $m^2(x-y)-m^2(x+y)+m^2(y+z)+m^2(y-z)$.
36. $3[x-(y+z)]-3[y-(x+z)]-6(x-y+z)$.

Given $A = x + y$, $B = x - y$, $C = x^2 + xy + y^2$, *and* $D = x^2 - xy + y^2$, *find:*

37. $C + D$.
38. $C - D$.
39. $D - C$.
40. $C + 2D$.
41. AB.
42. BA.
43. AC.
44. BC.
45. AD.
46. BD.
47. DC.
48. CD.
49. ABC.
50. ABD.
51. ACD.
52. BCD.

53. Add $ax^2 + bx + c$ and $mx^2 + nx + p$.

54. From $am^2 + bm + c$ subtract $pm^2 + qm + r$.

55. Add ax, bx, cx, dx, $-ax$, $2ax$, x, and $-x$.

56. Add abx, $-cdx$, $2abx$, $3cdx$, 4, and $-2cdx$.

57. From the sum of ax and bx subtract $(a - b)x$.

58. From the sum of a^2x and $-b^2x$ subtract $-b^2x - a^2x$.

59. If 1 more than a certain number is multiplied by the number, the product equals 3 more than the square of the number. What is the number?

60. If 2 less than a certain number is multiplied by the number, the product equals 4 less than the square of the number. What is the number?

61. If a certain number increased by 1 is multiplied by the number increased by 2, the product equals the sum of 8 and the square of the number. What is the number?

62. If a certain number increased by 2 is multiplied by the number increased by 3, the product equals 16 more than the square of the number. What is the number?

63. If the number of students in a certain algebra class is increased by 4, and this sum is multiplied by the number increased by 5, the product equals 200 more than the square of the number. What is the number?

64. By multiplying $a + b$ by $a + b$ show that the square of the sum of two numbers equals the square of the first plus twice the product of the first and second, plus the square of the second. Apply this law to squaring $20 + 5$.

CHAPTER VII

DIVISION

Exercise 59. Division of Monomials

Examples 1 to 18, oral — Examples 19 to 34, written

1. Divide by 2: 4 mi.; $4m$; $4 \cdot 5$; $4x$; $4 \cdot 3 \cdot 5$; $4xy$.

2. Divide by 5: 10 lb.; $10l$; $10n$; $10n^2$; $10xy$; $10abc$.

3. Divide by 9: 27 rd.; $27r$; 36 in.; $36i$; 45 acres; $45a$.

4. If the temperature is 4° below zero, what will it be when it is half as much below zero? How much is $-4° \div 2$?

5. 64¢ ÷ 8.
6. $64c \div 8$.
7. $64xy \div 8$.
8. $64\sqrt{a} \div 8$.
9. $-4° \div 2$.
10. $-4x \div 2$.
11. $-4abc \div 2$.
12. $-16x^2 \div 2$.
13. $75ab \div 25$.
14. $60a^2x \div 30$.
15. $45p^2q^2 \div 15$.
16. $90\sqrt{x} \div 45$.

17. How do you divide a monomial by a positive integer?

18. Divide \$8 by \$2; $8d$ by $2d$; 8 ft. by 2 ft.; $8f$ by $2f$; $8 \cdot 5$ by $2 \cdot 5$; 8 times any number by 2 times the number.

Divide:

19. $8x$ by $2x$.
20. $10x$ by $2x$.
21. $15y$ by $5y$.
22. $16z$ by $8z$.
23. $25m$ by $5m$.
24. $36p$ by $4p$.
25. $40a$ by $10a$.
26. $75b$ by $25b$.
27. $125a^2$ by $5a^2$.
28. $275x^2$ by $25x^2$.
29. $250p^2$ by $50p^2$.
30. $175m$ by $25m$.
31. $-350ab$ by $70ab$.
32. $-450pq$ by $25pq$.
33. $-750a^2x$ by $25a^2x$.
34. $-925abc$ by $25abc$.

72. Law of Exponents. Division being the inverse of multiplication, we have the following:

Since $a^2 \cdot a^3 = a^5$, therefore $a^5 \div a^3 = a^2$;

since $a^x \cdot a^y = a^{x+y}$, therefore $a^{x+y} \div a^y = a^x$,

or $a^m \div a^n = a^{m-n}$.

The exponent of any letter in the quotient is equal to the exponent of that letter in the dividend minus the exponent of that letter in the divisor.

That is, $a^m \div a^n = a^{m-n}$.

Likewise, $-21x^9y^4 \div 7x^5y = -3x^4y^3$.

In algebra, division is usually expressed in the fractional form, thus:

$$\frac{48a^5b^4c^3}{-4abc} = -12a^4b^3c^2.$$

73. Law of Signs. The Law of Signs is given on page 41. Briefly stated it is as follows:

In division, two like signs produce plus; two unlike signs produce minus.

That is, $\frac{4}{2} = 2$, $\frac{-4}{-2} = 2$, $\frac{-4}{2} = -2$, and $\frac{4}{-2} = -2$.

Exercise 60. Division of Monomials

Examples 1 to 5, oral — Examples 6 to 20, written

1. $\frac{a^2b^2}{ab}$.
2. $\frac{m^8n^4}{mn}$.
3. $\frac{x^7y^5}{xy}$.
4. $\frac{x^9y^8}{x^2y^2}$.
5. $\frac{-4a^5}{2a^3}$.
6. $\frac{11.1x^7}{3x^2}$.
7. $\frac{-9.6m^4n}{-0.4mn}$.
8. $\frac{135a^4b^8c}{5abc}$.
9. $\frac{96x^4y^4z^4}{-6xyz}$.
10. $\frac{1.21p^5q^5r}{1.1pq^2r}$.
11. $\frac{-119m^7n^5}{7m^5n^5}$.
12. $\frac{144m^4n^6}{-4m^4n}$.
13. $\frac{-175x^2y^4z^6}{-5x^2yz^5}$.
14. $\frac{-225m^2n^2p^4}{-25mn^2p}$.
15. $\frac{375a^{10}b^9c^8}{25ab^7c^3}$.
16. $\frac{-425p^8q^7r^8}{5p^6q^7r}$.
17. $\frac{603a^8b^7c^6}{-9ab}$.
18. $\frac{-504x^{10}y^9z}{3x^9y^8}$.
19. $\frac{25a^mb^mc^m}{-5abc}$.
20. $\frac{-36a^mb^mc^m}{-4a^nb^nc^n}$.

74. Division of a Polynomial by a Monomial. If we divide 10 ft. 8 in. by 2, we have 5 ft. 4 in. Similarly, we have

$$\begin{array}{lll}
2\underline{)10\text{ ft. }4\text{ in.}} & 4\underline{)40\text{ yd. }16\text{ in.}} & 9\underline{)9\text{ tens}+9} \\
\quad 5\text{ ft. }2\text{ in.} & \quad 10\text{ yd. }\ 4\text{ in.} & \quad 1\text{ ten }+1 \\
2\underline{)10f+4i} & 4\underline{)40\cdot 3+16\cdot 5} & 9\underline{)9t+9} \\
\quad 5f+2i & \quad 10\cdot 3+\ 4\cdot 5 & \quad t+1
\end{array}$$

That is, to divide a polynomial by a monomial,

Divide each term of the dividend by the divisor and add the partial quotients.

If we divide zero by any number, the result is zero. Division by zero has as yet no meaning; the expression $a \div 0$ will be discussed later.

Exercise 61. Division of a Polynomial by a Monomial

Examples 1 to 9, oral — Examples 10 to 17, written

1. Divide by 2: 8 ft. 6 in.; $8f+6i$; 8 tens + 6; $8t+6$; 86.
2. Divide by 3: 9 mi. 15 rd.; $9m+15r$; $9xy+15xy$.
3. Divide by a: $ax+ay$; $abc+axy$; $a\sqrt{x}+a\sqrt{y}$; $a+a^2$.

Divide:

4. px^2+py^2 by p.
5. $axy+amn$ by a.
6. $mpq-mxy$ by m.
7. $-ax+ay$ by a.
8. $-ax+ay$ by $-a$.
9. $am^2-an^2+ax^2$ by a.

Perform the indicated divisions:

10. $\dfrac{35p^4m+75p^2m^3}{5p^2m}$.
11. $\dfrac{64a^2b^3c^4-48a^4b^3c^2}{-4a^2b^2c^2}$.
12. $\dfrac{75a^5b^4c^3-65a^3b^4c^5}{-5a^3b^3c^3}$.
13. $\dfrac{-81m^4n^3+108m^3n^4}{-9m^3n^3}$.
14. $\dfrac{25m^3p+35m^4q-15m^5r}{-5m^3}$.
15. $\dfrac{ab^2c^3-a^3b^2c+a^2b^2c^2}{ab^2c}$.
16. $\dfrac{-4ab^2x-6a^2by+8abxy}{-2ab}$.
17. $\dfrac{-9mnx^3+15m^2n^2x^2-3mn}{-3mn}$.

75. Division of a Polynomial by a Polynomial. Since divisio is the inverse of multiplication, the process of division is be understood by first studying a case in multiplication.

Multiplicand	$a^2 + 2ab + b^2$
Multiplier	$a + b$
1st partial product, by a	$a^3 + 2a^2b + ab^2$
2d partial product, by b	$a^2b + 2ab^2 + b^3$
Product by $a + b$	$a^3 + 3a^2b + 3ab^2 + b^3$

Reversing this, required to divide $a^3 + 3a^2b + 3ab^2 + b^3$ by $a^2 + 2ab + b^2$. The work is more conveniently arranged by placing the divisor at the right, thus:

Dividend	$a^3 + 3a^2b + 3ab^2 + b^3$	$a^2 + 2ab + b^2$	Divisor
Product by a	$a^3 + 2a^2b + ab^2$	$a + b$	Quotient
Remainder	$a^2b + 2ab^2 + b^3$		
Product by b	$a^2b + 2ab^2 + b^3$		

Dividing a^3, the first term of the dividend, by a^2, the first term of the divisor, we have a, the first term of the quotient.

Multiplying the divisor by a, we have $a^3 + 2a^2b + ab^2$, which is subtracted from the dividend, leaving $a^2b + 2ab^2 + b^3$ still to be divided.

Proceeding as before, and dividing a^2b by a^2, the next term of the quotient is b.

Multiplying as before, we have $a^2b + 2ab^2 + b^3$, which is subtracted, leaving no remainder.

We therefore see that in division we proceed as follows:

Arrange both dividend and divisor in ascending or descending powers of some common letter.

Divide the first term of the dividend by the first term of the divisor and write the result for the first term of the quotient.

Multiply the entire divisor by the first term of the quotient and subtract the result from the dividend.

If there is a remainder, consider it as a new dividend and proceed as before.

76. Illustrative Problems. The following typical problems should be carefully studied:

1. Divide $x^2 - 5x - 84$ by $x + 7$.

OPERATION

$$\begin{array}{r|l} x^2 - 5x - 84 & x + 7 \\ \cline{2-2} \underline{x^2 + 7x \phantom{{}-84}} & x - 12 \\ -12x - 84 & \\ \underline{-12x - 84} & \end{array}$$

CHECK

$$\frac{-88}{8} = -11$$

We may check the result (1) by carefully reviewing the work, (2) by multiplying the quotient by the divisor, the product being the dividend, or (3) by substituting some convenient values for the letters.

Applying the second of these checks, we can easily show that $(x + 7)(x - 12) = x^2 - 5x - 84$.

Applying the third check, letting $x = 1$, we have $1 - 5 - 84 = -88$, $1 + 7 = 8$, $-88 \div 8 = -11$; and the quotient, $x - 12$, becomes -11 also.

2. Divide $a^3 - b^3$ by $a - b$.

OPERATION

$$\begin{array}{l|l} a^3 - b^3 & a - b \\ \cline{2-2} \underline{a^3 - a^2b} & a^2 + ab + b^2 \\ \quad a^2b - b^3 & \\ \quad \underline{a^2b - ab^2} & \\ \qquad ab^2 - b^3 & \\ \qquad \underline{ab^2 - b^3} & \end{array}$$

CHECK

$a = 2, b = 1$

$$\frac{7}{1} = 7$$

In the check we cannot let $a = b$ because this would make the divisor zero. We therefore let $a = 2$ and $b = 1$. Then the dividend is 7, the divisor 1, and the quotient 7.

Since we cannot divide by zero, in checking the work we use some value for the letters that shall not make the divisor zero.

It is also a good check to notice that if both dividend and divisor are homogeneous, the quotient is also homogeneous. In the case given above the dividend, divisor, and quotient are all homogeneous.

Exercise 62. Division of a Polynomial by a Polynomial

Examples 1 to 5, oral — Examples 6 to 29, written

1. In dividing $x^3 + 3x^2y + 2xy^2$ by $x^2 + 2xy$, what is the first term of the quotient?

2. How do you check the work in division?

State the first term of the quotient:

3. $x^2 - 5x + 6$ divided by $x - 3$; by $x - 2$.

4. $8m^2 - 10mn - 3n^2$ divided by $4m + n$; by $2m - 3n$.

5. $p^3 - 8p - 3$ divided by $-p + 3$; by $p^2 + 3p + 1$.

Divide the following, checking the results:

6. $a^2 + 3a + 2$ by $a + 1$.

7. $b^2 + 5b + 4$ by $b + 1$.

8. $c^2 + 8c + 12$ by $c + 2$.

9. $d^2 - 2d - 15$ by $d + 3$.

10. $e^2 + 11e + 28$ by $e + 4$.

11. $f^2 - 3f - 40$ by $f + 5$.

12. $g^2 - 11g + 18$ by $g - 2$.

13. $h^2 - 13h + 30$ by $h - 3$.

14. $a^2 - 5ab - 6b^2$ by $a + b$.

15. $a^2 - 6ab + 5b^2$ by $a - b$.

16. $c^2 + cd - 6d^2$ by $c + 3d$.

17. $p^2 - 9pq + 20q^2$ by $p - 5q$.

18. $x^2 + xy - 20y^2$ by $x - 4y$.

19. $p^2 + pq - 20q^2$ by $p + 5q$.

20. $x^2 + 11xy + 28y^2$ by $x + 7y$.

21. $p^2 + 11pq + 28q^2$ by $p + 4q$.

Arrange according to the descending powers of x, and divide:

22. $x^3 + x^4 - 16x - 4 - 9x^2$ by $4 + x^2 + 4x$.

23. $6 + x^4 - 12x + 11x^2 - 5x^3$ by $3 - 3x + x^2$.

24. $2x^4 + 9x - 12 - 5x^3 - 7x^2$ by $1 - x + x^2$.

25. $-7x^3 - 10x^2 + x^4 - 3 + 25x$ by $x + x^2 - 3$.

26. $3x^3 + x^4 - 6x - 4x^2 + 4$ by $3x + x^2 - 2$.

27. $6x - x^2 + x^4 - 9$ by $x^2 + 3 - x$; by $x^2 + x - 3$.

28. If $f(x) = x^2 + 7x + 2$ and $F(x) = x^3 + 8x^2 + 9x + 2$, find $F(x) \div f(x)$; $F(x) \div (x + 1)$; $(f(x) + 10) \div (x + 3)$.

29. Divide $2h + 8t + 8u$ by $h + 4t + 4u$; 288 by 144.

77. Fraction in the Quotient. In algebra as in arithmetic, the quotient may contain a fraction.

For example, divide $a^3 + b^3$ by $a - b$.

OPERATION

$$\begin{array}{rrr|l}
a^3 & + & b^3 & a-b \\ \cline{4-4}
a^3 - a^2b & & & a^2+ab+b^2+\dfrac{2b^3}{a-b} \\ \cline{1-3}
a^2b + & & b^3 & \\
a^2b - ab^2 & & & \\ \cline{1-3}
ab^2 + & & b^3 & \\
ab^2 - & & b^3 & \\ \cline{1-3}
 & & 2b^3, & \text{remainder}
\end{array}$$

CHECK

$a = 2, b = 1$

$\frac{9}{1} = 7 + \frac{2}{1} = 9$

Here we have a remainder of $2b^3$. If we continue the division, the next term of the quotient becomes $\frac{2b^3}{a}$, and all of the other terms are fractional, the fractions being written precisely as in arithmetic. We therefore simply express the division by writing the remainder over the whole divisor, thus: $\frac{2b^3}{a-b}$. The quotient is therefore $a^2 + ab + b^2 + \frac{2b^3}{a-b}$.

The subject of fractions is treated later. For the present we may write them as in arithmetic, except that in algebra we write the proper sign between an integer and a fraction, as in $a + \frac{b}{c}$.

Exercise 63. Fraction in the Quotient

Written Work

1. Divide $x^3 + 1$ by x; by $x + 1$; by $x - 1$.
2. Divide $x^4 + 1$ by x^2; by x^4; by $x + 1$; $x^3 - 1$ by $x + 1$.
3. Divide $x^2 + 2x + 2$ by $x + 1$; by $x - 1$; by $x + 2$.

Divide the following:

4. $x^2 + 3x + 1$ by $x - 1$.
5. $x^2 - 4x + 2$ by $x - 2$.
6. $x^2 - 2x - 5$ by $x - 3$.
7. $x^2 + 7x + 4$ by $x - 2$.
8. $a^2 + 2a - 7$ by $a + 1$.
9. $a^2 + 3a - 8$ by $a + 2$.
10. $m^2 + 4m - 4$ by $m + 4$.
11. $p^2 + 5pq + q^2$ by $p + q$.

Exercise 64. Equations involving Division

Examples 1 to 11, oral — Examples 12 to 32, written

1. If $7x = 63$, what is the value of x? How do you prove it?

2. If $ax = 3a$, what is the value of x? Prove it.

3. If $2a^2x = 4a^4$, what is the value of x? Prove it.

Find the value of x and prove that it is correct:

4. $3ax = 6a$.

5. $5a^2x = 10a^2$.

6. $7abx = 14a^2b^2$.

7. $9abx = 27a^3b^3$.

8. $-2ax = 24a^2$.

9. $4ax = -16a^3$.

10. $-7a^nx = -56a^n$.

11. $-abcx = -a^2b^2c^2$.

12. If $(a+b)x = a^2 + 2ab + b^2$, what is the value of x?

Find the value of x and prove that it is correct:

13. $ax = a^3 + a^2b$.

14. $a^4x = a^9 - 3a^5$.

15. $a^7x = 3a^{17} - a^7$.

16. $-ax = -a^2 + a$.

17. $-17ax = -357a^m$.

18. $(a-b)x = a^3 - b^3$.

19. $(a+b)x = a^3 + b^3$.

20. $(a-7)x = a^3 - 343$.

21. $(a+7)x = a^3 + 343$.

22. $(a^2 - b^2)x = a^4 - b^4$.

Given the following polynomials:

$M = a^2 + 2ab + b^2$, $P = a^3 + 3a^2b + 3ab^2 + b^3$,

$N = a^2 - 2ab + b^2$, $Q = a^3 - 3a^2b + 3ab^2 - b^3$,

find the value of x in the following:

23. $Mx = P$.

24. $Mx = -P$.

25. $Mx = -3P$.

26. $Nx = Q$.

27. $Nx = -Q$.

28. $Nx = -3Q$.

29. $(a-b)x = Q$.

30. $(a+b)x = P$.

31. $MNx = PQ$.

32. If $i = prt$, what is the value of t in terms of i, p, and r? Evaluate the result if $i = 27$, $p = 450$, and $r = 6\%$; if $i = 162$, $p = 450$, and $r = 6\%$.

CHAPTER VIII

SIMPLE EQUATIONS WITH ONE UNKNOWN QUANTITY

78. Simple Equation. An equation involving the first power and no higher power of the unknown quantity is called a *simple equation.*

Such an equation is also said to be of the *first degree.* It is also called, particularly in higher mathematics, a *linear equation.*

79. Identity. An equation that is true for any value whatsoever of any letter is called an *identity.*

Thus $(a+b)^2 = a^2 + 2ab + b^2$. This is true if $a = 3$, $b = 4$, or if the letters have any other values whatsoever.

An equation involving only numbers, like $2 + 3 = 5$, is also called an identity.

An identity is sometimes expressed by the symbol $\equiv$, as in $a + b \equiv b + a$, read "$a + b$ is identical to $b + a$."

80. Satisfying an Equation. The quantity that, substituted for the unknown quantity, reduces an equation to an identity is said to *satisfy* the equation.

This quantity is the value of the unknown quantity. Thus if $x + 7 = 9$, then $x = 2$, and 2 satisfies the equation.

If $ax = a^2$, and we consider x as the unknown quantity, we see that $x = a$, and a satisfies the equation.

81. Solving an Equation. To find the value or values of the unknown quantity that will satisfy the equation is to *solve* the equation.

Thus if $x + 3 = 5$, then $x = 2$; for if we put 2 for x the two members of the equation become $2 + 3$, and 5, which are the same in value. In other words, the equation then becomes an identity.

82. Root. A value of the unknown quantity that satisfies an equation is called a *root* of the equation.

Thus in $x - 4 = 9$, 13 is the root of the equation.

Simple equations have only one root. We shall later study equations that have more than one root. For example, the equation $x^2 - 3x + 2 = 0$ is satisfied if $x = 1$, and also if $x = 2$.

83. Transposition. We have already learned that we may subtract equals from equals and have equals left (§ 19). We have also seen that this amounts to taking a quantity from one side of the equation, changing the sign, and putting it on the other side.

Taking a quantity from one side of an equation, changing its sign, and putting it on the other side is called *transposition.*

For example, if $2x - 4 = 8 + x$,

we may subtract x from both sides and have

$$x - 4 = 8.$$

We may then subtract -4 from both sides (or add $+4$ to both sides) and have $x = 12$.

The word "transpose" is therefore unnecessary, since we can use "subtract" or "add" in its stead, but it is a word that is commonly employed by many teachers of algebra.

84. Numerical Equation. An equation containing no letters except those representing a root is called a *numerical equation.* The following is the method of solving such an equation.

Solve the equation $4x - 7 + x = 10 - 3x - 1$.

Combining the terms, $5x - 7 = 9 - 3x$.

Subtracting -7, $5x = 16 - 3x$.

Subtracting $-3x$, $8x = 16$.

Dividing by 8, $x = 2$.

Check. $4 \times 2 - 7 + 2 = 10 - 3 \times 2 - 1$,

or $3 = 3$.

The solution may be shortened, thus:

Combining terms, $5x - 7 = 9 - 3x$.

Adding 7 and $3x$, $8x = 16$.

Dividing, $x = 2$.

Exercise 65. Numerical Equations

Examples 1 to 15, oral — Examples 16 to 32, written

1. Is $x + 3 = 4$ an identity? Give your reason.

2. Is $x + 3 = 3 + x$ an identity? Give your reason.

3. What value of x satisfies the equation $x - 4 = 5$? Prove it.

4. What is the root of the equation $x + 7 = 12$? Prove it.

5. What right have you to transpose -3 in the equation $x - 3 = 9$? Explain why the sign is changed.

Solve the equations:

6. $x - 4 = 6$.
7. $x + 4 = 6$.
8. $6 - x = 4$.
9. $x + 5 = 9$.
10. $x - 5 = 9$.
11. $9 - x = 5$.
12. $x + 7 = 15$.
13. $x + 15 = 7$.
14. $x - 15 = 7$.

15. How will you begin to solve the equation $4x + 5 - 2x = 15$? What will be your second step? your third step?

Solve the equations:

16. $5x - 7 + 3x = 9$.
17. $7x - 4 - 5x = 10$.
18. $9x + 7 - 37 = 3x$.
19. $9x - x - 44 = 4x$.
20. $7 - x + 7x = 31$.
21. $8 + 2x - 19 = x$.
22. $5 - x - 8 + 6x = 17$.
23. $3 - 7x - 14x - 1 = 23$.
24. $5x - 7 + 7x - 5 = 36$.
25. $9\frac{1}{2}x + 7 + 3\frac{1}{2}x = 20$.

26. If to 5 times a certain number we add 2, the sum is 20 diminished by 4 times the number. What is the number?

Solve the equations:

27. $5x - 7 + 3x + 6 + 8x = 13 + 6x + 40$.
28. $7x - 2 + 9x - 3 + 5x = 4 + x + 31$.
29. $8x + 7 - 2x + 5 = 4x + 12 - (x - 30)$.
30. $5x - (4 - 9x) - 6 = 2x - (7 - 3x) + 15$.
31. $3\frac{2}{3}x + 2\frac{1}{2} - (5\frac{1}{8}x - 7\frac{3}{4}) = 7\frac{1}{8}x - (5\frac{3}{4} - 9\frac{2}{3}x) + 26\frac{1}{4}$.
32. $2\frac{7}{8}x + 3\frac{3}{4} + (5\frac{1}{4}x - 9\frac{1}{2}) - 12\frac{5}{8}x = 2\frac{1}{4}x + 5\frac{3}{8} - (7\frac{3}{4}x - 12)$.

85. Problems relating to Numbers. The following problems should be studied carefully before attempting the next exercise.

1. Five times a certain number equals 75. Find the number.

Let $x =$ the number.

Then $5x =$ five times the number.

But $75 =$ five times the number.

$\therefore 5x = 75.$

Dividing by 5, $x = 15.$

Therefore the required number is 15.

Check. Substituting in the statement of the problem, $5 \times 15 = 75.$

2. Five times a certain number equals the number increased by 24. Find the number.

Let $x =$ the number.

Then $5x =$ five times the number,

and $x + 24 =$ the number increased by 24.

$\therefore 5x = \cancel{x} + 24.$

Subtracting x, $4x = 24.$

Dividing by 4, $x = 6.$

Therefore the required number is 6.

Check. Substituting in the statement of the problem, $5 \times 6 = 6 + 24.$

3. The sum of two numbers is 39, and one of them is three more than five times the other. Find the numbers.

Let $x =$ the smaller number.

Then $5x + 3 =$ the larger number.

Since their sum is 39,

$$(5x + 3) + x = 39.$$

Combining, $6x + 3 = 39.$

Subtracting 3, $6x = 36.$

Dividing by 6, $x = 6.$

Then
$$\begin{aligned} 5x + 3 &= 5 \times 6 + 3 \\ &= 33. \end{aligned}$$

Therefore the required numbers are 6 and 33.

Check. Substituting in the statement of the problem,

$$6 + 33 = 39,$$

and
$$33 = 5 \times 6 + 3.$$

After a little practice solutions of this nature may be shortened.

Exercise 66. Problems in Numerical Equations

Examples 1 to 13, oral — Examples 14 to 29, written

1. If you represent a number by x, how will you represent four times the number? five more than four times the number?

2. If one book costs x dollars, how much will seven books cost? nine books? fifteen books? n books?

3. If a man bought 15 horses at d dollars each, how much did he pay for all the horses? for a third of them?

4. How will you represent four times a number, increased by seven? four times a number, decreased by six?

5. How will you represent the excess of a number over 45? the excess of 45 over a number? a number increased by 45?

6. Express in cents the value of d dimes; of q quarters; of n dimes plus q quarters; of d dimes and c cents.

7. If you are n years old to-day, how old were you five years ago? How old will you be in seven years?

8. A gallon contains 231 cu. in. How many cubic inches are there in g gallons? in $2g$ gallons? in $\frac{1}{3}g$ gallons?

9. If twice a certain number is 36, what is the number?

10. If half a certain number is 24, what is the number?

11. If the sum of 7 and a certain number is 20, what is the number? What is twice the number?

12. If a certain number is 5 more than 7, what is the number? What is half the number?

13. If twice a certain number, decreased by 1, is 9, find the number. Check the result.

14. Four increased by three times a certain number equals 19. Find the number.

15. Four times a certain number is diminished by three, the result being 29. Find the number.

16. If from five times a certain number seven is subtracted, the result is 33. Find the number.

17. Twelve more than three times a certain number equals 45. Find the number.

18. Eight times a certain number equals 27 diminished by the number. Find the number.

19. Nine times a certain number equals 75 diminished by six times the number. Find the number.

20. Twelve times a certain number equals 80 diminished by eight times the number. Find the number.

21. Fifteen times a certain number equals 58 diminished by fourteen times the number. Find the number.

22. Nine times a certain number is diminished by 3.42, the result being 77.58. Find the number.

23. Ten times a certain number is diminished by six, the result being 36 more than four times the number. Find the number.

24. Twelve times a certain number is diminished by nine, the result being 54 more than nine times the number. Find the number.

25. Six times a certain number is increased by 15, the result being 45 more than four times the number. Find the number.

26. Seventeen times a certain number is increased by 13, the result being 77 more than nine times the number. Find the number.

27. Twenty-three times a certain number is increased by five times the number, the result being 160 more than eight times the number. Find the number.

28. From $5\frac{2}{3}$ times a certain number 7 is subtracted, the result being 93 more than $2\frac{1}{3}$ times the number. Find the number.

29. From $7\frac{3}{8}$ times a certain number $9\frac{1}{3}$ is subtracted, the result being $28\frac{13}{24}$ more than $5\frac{1}{4}$ times the number. Find the number.

86. Problems relating to Per Cents. The following problems are typical of those met in percentage.

1. A man receives 6% on some money invested and adds \$60 to the amount received, thus making \$300. How much has he invested?

Let $x =$ the number of dollars invested.

Then $0.06x =$ the number of dollars received,

and $0.06x + 60 =$ the number of dollars stated, or 300.

$\therefore 0.06x + 60 = 300.$

Subtracting 60, $0.06x = 240.$

Dividing by 0.06, $x = 4000.$

Therefore he has \$4000 invested.

Check. 6% of \$4000 = \$240, and \$240 + \$60 = \$300.

It should be observed that we do not let x equal the *money*, but we let $x =$ the *number* of dollars. Then when we find that $x = 4000$, we know that this is the *number* of dollars, and that \$4000 is the sum invested. We are thus relieved of the necessity of considering the dollar sign in the solution. In general, in algebra, this plan is pursued, all the numbers being considered abstract.

2. What per cent above cost must a man mark his goods so as to allow a discount of 20% and still make a profit of 20%?

The question is to find what per cent the marked price is of the cost, and then to find how much this is above cost.

Let $c =$ the number of dollars of cost.

Then $1.20c =$ the number of dollars of selling price.

Let $m =$ the number of dollars of marked price.

Then $0.80m =$ the number of dollars of selling price.

Therefore $0.80m = 1.20c.$

Dividing by 0.80, $m = 1.50c.$

That is, the goods must be marked 50% above cost.

Check. If from $1.50c$ we take 20% of it, we have left 80% of $1.50c$, or $1.20c$. This $1.20c$ is 20% more than c.

For example, if the goods cost \$80 the dealer marks them at 1.50 × \$80, or \$120. The selling price is then 20% less, or \$96, and this is 20% more than \$80.

Exercise 67. Problems relating to Per Cents

Examples 1 to 12, oral — Examples 13 to 20, written

1. Six is six per cent of what number?

2. Three is six per cent of what number?

3. Six is three per cent of what number?

Solve:

4. $6\%\, x = 18.$
5. $5\%\, x = 15.$
6. $4\%\, x = 12.$
7. $3\%\, x = 9.$
8. $2\%\, x = 6.$
9. $12\%\, x = 24.$
10. $x + 6\%\, x = 106.$
11. $x + 5\%\, x = 105.$
12. $x + 5\%\, x = 210.$

13. If 18% by weight of wheat is lost in grinding it into flour, how many pounds of wheat are used if the loss is 360 lb.?

14. From Ex. 13, how many pounds of wheat are needed to produce 820 lb. of flour?

15. From Ex. 13, how many pounds of wheat are needed to produce 779 lb. of flour?

16. A dealer sells a suit of clothes for $32.20. The suit cost him $28. What is his per cent of gain?

17. A suit of clothes was sold for $30.80, after a discount of 12% was allowed on the marked price. What was the marked price?

18. The profits on a business this year are 22% more than they were last year. This year they are $6344. What were they last year?

19. Water in freezing expands 10% of its volume. How many cubic inches of water will make 508.2 cu. in. of ice? Allowing 231 cu. in. to a gallon, how many gallons of water are needed?

20. At what per cent above cost must a dealer mark his goods so that he can deduct 25% and still make a profit of 25%? What will be the marked price if the goods cost the dealer $800? What will be the selling price?

87. Problems relating to Mixtures. In various kinds of business it becomes necessary to mix certain ingredients, and the quantities are often found easily by algebra.

How much water must be added to 12 qt. of a 25% solution of ammonia to reduce it to a 10% solution?

Let $x =$ the number of quarts added.
Then $12 + x =$ the total number of quarts.
Then $10\%(12 + x) =$ the number of quarts of ammonia.
But 25% of $12 =$ the number of quarts of ammonia.
$\therefore 10\%(12 + x) = 25\%$ of 12.
Combining, $1.2 + 10\%x = 3$.
Subtracting 1.2, $10\%x = 1.8$.
Dividing by 10%, $x = 18$.
Therefore 18 qt. of water must be added.

Check. 25% of 12 qt. = 3 qt., the amount of ammonia.
12 qt. + 18 qt. = 30 qt., the amount of the mixture.
Furthermore, 3 qt. is 10% of 30 qt., as required.

Exercise 68. Problems relating to Mixtures

Examples 1 to 5, oral — Examples 6 to 17, written

1. In the equation $5\%(1 + x) = 20\%$, what is the first step in the solution? the second step?

2. If we add x qt. to 4 qt. and take 10% of the sum, how shall we indicate the result?

3. If a certain ore yields 2% pure metal, how many pounds will it yield to the ton (2000 lb.)?

4. If a certain ore yields 60 lb. of pure metal to the ton, what per cent does it yield?

5. If 20 lb. of tin is melted with 60 lb. of copper, what part of the mixture is tin? copper? What per cent of the mixture is tin? copper? The tin is what per cent of the weight of the copper? The copper is what per cent of the weight of the tin?

6. How much water must be added to a gallon of a 20% solution of ammonia to reduce it to a 10% solution?

7. How much water must be added to 32 gal. of alcohol 95% pure to reduce it to a 75% solution?

8. How many pounds of water must be added to 160 lb. of a 5% solution of salt to reduce it to a 4% solution? How many gallons, allowing 8.35 lb. of water to a gallon?

9. How much water must be added to a pint of a 5% solution of a certain medicine to reduce it to a 1% solution?

10. How much water must be added to a barrel of vinegar 85% pure to reduce it to a 50% solution?

11. How much vinegar must be added to a barrel of vinegar 50% pure to make it an 85% solution?

12. How much ammonia must be added to a gallon that contains 20% pure ammonia so that the mixture shall contain 30% pure ammonia?

13. In an alloy of 90 oz. of silver and copper there are 6 oz. of silver. How much copper must be added that 50 oz. of the new alloy may contain 2 oz. of silver?

14. In a mixture of 3 oz. of water and listerine there is 1 oz. of listerine. How much listerine must be added to make the new solution contain 75% pure listerine?

15. Our silver coins are 90% pure silver. How much pure silver must be melted with 100 oz. of silver alloy containing 80% pure silver to bring it to the standard required for our coinage?

16. In Ex. 15 how much pure silver must be melted with 200 oz. of silver alloy containing 75% pure silver to bring it to the 90% standard?

17. Air consists of five parts of nitrogen and one part of oxygen. How many cubic feet of oxygen are there in a schoolroom 40 ft. by 30 ft. by 10 ft.? How high would a room 40 ft. by 30 ft. be if it contained 2400 cu. ft. of oxygen?

88. Problems relating to Motion. Algebra is useful in solving problems relating to motion.

1. A train leaves Pittsburgh for the West at the same time that one leaves for the East. The former travels at the average rate of 42 mi. an hour and the latter at the rate of 38 mi. an hour. In how many hours will they be 240 mi. apart?

Let $x =$ the number of hours.

In 1 hr. they are $(42 + 38)$ mi. apart, or 80 mi. apart.

In x hr. they are $x \cdot 80$ mi. apart, or $80x$ mi.

But they are 240 mi. apart.

$$\therefore 80x = 240.$$

Dividing by 80, $x = 3.$

Therefore in 3 hr. they will be 240 mi. apart.

Check. $3 \times (48 + 32)$ mi. $= 240$ mi.

2. A man starts from a certain place and travels on his bicycle at the rate of 16 mi. an hour. Forty-five minutes later an automobile starts after him at the rate of 24 mi. an hour. How long will it take the automobile to overtake him?

In 45 min. the bicycle has gone $\frac{3}{4}$ of 16 mi., or 12 mi.

The automobile goes 24 mi. an hour while the bicycle goes 16 mi., the automobile thus gaining on the bicycle 8 mi. an hour.

Let $x =$ the number of hours required.

Since the automobile gains 8 mi. an hour,

$8x =$ the number of miles gained in x hr.

But $12 =$ the number of miles to be gained.

$$\therefore 8x = 12.$$

Dividing by 8, $x = 1\frac{1}{2}.$

Therefore the automobile will overtake the bicycle in $1\frac{1}{2}$ hr.

Check. In $1\frac{1}{2}$ hr. the automobile will travel $1\frac{1}{2} \times 24$ mi., or 36 mi., and the bicycle will travel $1\frac{1}{2} \times 16$ mi., or 24 mi. Adding 12 mi., the start of the bicycle, 24 mi. + 12 mi. = 36 mi.

In all problems relating to motion the average rate is to be understood unless the contrary is stated. In other words, the motion referred to in such problems is to be taken as uniform.

Exercise 69. Problems relating to Motion

Examples 1 to 4, oral — Examples 5 to 18, written

1. At 3 mi. an hour, how far will a man travel in 4 hr.? in $\frac{1}{3}$ hr.? in t hours? in $(a + b)$ hours?

2. At r miles an hour, how far will an aëroplane go in t hours? in $2t$ hours? in $\frac{1}{2}$ hr.?

3. If two men travel in opposite directions, one at the rate of 3 mi. an hour and the other at the rate of 40 mi. an hour, how far apart will they be in 1 hr.? in 2 hr.? in h hours?

4. If two men start from Chicago and New York at the same time and travel toward one another, the first at the rate of a miles an hour and the second at the rate of b miles an hour, how much nearer one another will they be in t hours?

5. Two men start from Denver at the same time, one traveling south 35 mi. an hour, and the other north 38 mi. an hour. How many miles apart will they be in 3 hr.? In how many hours will they be 292 mi. apart?

6. Two men start from the same place, one going east and the other going west, the former traveling twice as fast as the latter. In 4 hr. they are 300 mi. apart. Find the rate of each.

7. Two men start from the same place, one to the north and the other to the south, the former traveling 5 mi. an hour faster than the latter. In 2 hr. they are 150 mi. apart. Find the rate of each.

8. In Ex. 7 suppose the former travels 5 mi. an hour slower than the latter, and that in 2 hr. they are also 150 mi. apart. Find the rate of each.

9. A man leaves a friend at the railway station and starts to drive north just as his friend leaves on the train for the south. A half hour later they are 20 mi. apart. If the man drives one fourth as fast as the train travels, what is the rate of each?

10. Two men start from places 30 mi. apart and travel toward one another, the first at the rate of 7 mi. an hour and the second at the rate of 3 mi. an hour. How soon will they meet?

11. In Ex. 10 suppose the two men to travel at the same rate and to meet in 5 hr. Find the rate.

12. Two men start from places 600 mi. apart and travel toward one another, the first traveling 5 mi. an hour faster than the second. They meet in 8 hr. Find the rate of each.

13. A man starts on foot and walks at the rate of 3 mi. an hour. An hour after he has started a man sets out on horseback to overtake him and travels 6 mi. an hour. How soon will the second man overtake the first?

14. In Ex. 13 suppose the first man walks 4 mi. an hour and the second rides 6 mi. an hour. How many miles will the second man have to ride to overtake the first?

15. A train leaves St. Louis for Chicago at 8 A.M. and travels 35 mi. an hour. After an hour and a half a second train follows it on a parallel track at the rate of 47 mi. an hour. At what distance from St. Louis will the second train pass the first?

16. The distance from Albany to New York is 143 mi. A passenger train starts from New York at the same time that a freight train starts from Albany, and they travel toward one another. The passenger train makes 45 mi. an hour and the freight train 26½ mi. How soon will they meet and at what distance from New York?

17. In Ex. 16 suppose both trains to be going from Albany to New York, the freight having a start of 2 hr. How soon will the passenger train overtake the freight and at what distance from Albany?

18. In Ex. 16 suppose the trains travel 50 mi. and 25 mi. an hour, respectively, but the freight train is delayed 35 min. during the first hour. How soon will they meet?

89. Miscellaneous Problems. The following solutions should be studied before attempting the next exercise.

1. The sum of three consecutive numbers is 48. Find the numbers.

Let n = the middle number.

Then $n-1$ = the smallest number,

and $n+1$ = the largest number.

Adding, $3n$ = the sum.

Then $(n-1)+n+(n+1)=48,$

whence $3n=48.$

Dividing by 3, $n=16.$

Therefore the numbers are 15, 16, 17.

Check. $15+16+17=48.$

It simplifies the solution a little to take $n-1$, n, and $n+1$ for the numbers, instead of n, $n+1$, and $n+2$, although the latter plan is also correct. In this case we should have

$$n+(n+1)+(n+2)=48,$$

whence $3n=45,$

and $n=15.$

Therefore the numbers are 15, 16, 17.

2. A man is now three times as old as his son. Five years ago he was four times as old as his son. Find the age of each.

Let x = the number of years in the son's age.

Then $3x$ = the number of years in the father's age.

Also $x-5$ = the number of years in the son's age 5 years ago,

and $3x-5$ = the number of years in the father's age 5 years ago.

But five years ago the father was four times as old as the son.

$$\therefore 3x-5=4(x-5).$$

Multiplying, $3x-5=4x-20.$

Subtracting -5, $3x=4x-15.$

Subtracting $4x$, $-x=-15.$

Dividing by -1, $x=15$, the number of years in the son's age,

and $3x=45$, the number of years in the father's age.

Check. $45=3\times 15$; five years ago they were 40 years and 10 years of age respectively, and $40=4\times 10$.

Exercise 70. Miscellaneous Problems

Examples 1 to 6, oral — Examples 7 to 81, written

1. Solve the equations $5x + 7 = 42$; $5x - 7 = 43$.

2. Solve the equations $2x = 7$; $2x - 1 = 10$; $2x + 1 = 10$.

3. What number is one more than half itself?

4. A rectangle 5 in. wide has an area of 40 sq. in. What is its length?

5. A rectangular box 5 ft. long and 4 ft. wide has a capacity of 40 cu. ft. What is its depth?

6. A rectangular tank 5 ft. deep and 8 ft. wide has a capacity of 480 cu. ft. What is its length?

7. If it takes a steamer $5\frac{1}{2}$ da. to go 3355 mi., what is its average rate per day?

8. After playing 20 games a ball team had won three times as many as it had lost. How many games had it lost? How many had it won?

9. A freight train running 35 mi. an hour leaves a station three hours ahead of an express train that runs 50 mi. an hour. How soon will the express train overtake the freight?

10. The three angles of a triangle are together equal to 180°. In a certain right triangle one acute angle is three times as large as the other. Find the number of degrees in each angle.

11. The vertical angle of an isosceles triangle is 30°. Find the number of degrees in each of the two equal angles at the base.

12. A school building is in the form of a rectangle 60 ft. long and 49 ft. wide. An addition 35 ft. wide is to be made to cover half the area of the original building. How long is the addition?

13. If we can send a ten-word message to a certain place for 35¢, and a 24-word message for 63¢, what is the charge for each additional word above ten?

14. A miner secured 675 oz. of silver from some ore, this being $\frac{3}{4}\%$ of the weight of the ore. How much did the ore weigh?

15. The top of a tree 120 ft. high is broken off. The length of the part broken off is four times the length of the part left standing. Find the length of each part.

16. Three men buy a summer cottage for $3000. The second pays twice as much as the first, and the third pays as much as the first two together. How much does each pay? If they used the cottage for three months, each occupying it alone, how much of the time should each be allowed to use it?

17. In 30 years from now a boy will be three times as old as he is at present. How old is he now?

18. Ten years ago a boy was one third as old as he is at present. How old is he now?

19. Four years ago a man was seven times as old as his son, and his son is now eight years old. Find the age of the father.

20. A man has $9 in half dollars and quarters, having four times as many quarters as half dollars. How many coins of each kind has he?

21. A man has $3.85 in quarters and dimes, having three times as many dimes as quarters. How many coins of each kind does he have?

22. The sum of the ages of a father and his son is 60 years, and the father's age is three years more than twice the age of his son. Find the age of each.

23. Some boys are laying out a circular running track that is to be just a quarter of a mile in length. What radius must they use in describing the circle? (Use $\pi = 3\frac{1}{7}$, as in § 15.)

24. Ten men agreed to build a camp together, but two decided not to take their share, each of the others then having to pay $5 more for his share. Find the cost of the camp.

25. An ocean liner making 21 knots an hour leaves Boston when a freight steamer making 8 knots an hour is already 975 knots out. How long will it take the liner to overtake the freight steamer?

26. A river flows at the rate of 3 mi. an hour. A man who rows at the rate of 5 mi. an hour in still water wishes to row 1 mi. up the stream. How long will it take him?

27. A river flows at the rate of 2 mi. an hour. A man who rows at the rate of 5 mi. an hour in still water wishes to row 14 mi. down the stream. How long will it take him?

28. A river flows at the rate of 3 mi. an hour. A man rows a certain distance up the river in 12 hours and rows back to the starting place in 3 hours. If he keeps his regular rate in rowing, what is this rate in still water?

29. A man finds that it takes his naphtha launch two hours to go 24 mi. with the tide, and four hours to go 8 mi. against the tide. Find the rate of the tide in miles per hour.

30. A certain kind of solder is composed of $1\frac{7}{8}$ oz. of tin to 1 oz. of lead. How many pounds of each are required to make 46 lb. of solder?

31. The circumferences of the front and rear wheels of a wagon are respectively 2 yd. and 3 yd. How far must the wagon go for the front wheel to make one revolution more than the rear wheel? to make 50 revolutions more than the rear wheel? to make $2\frac{1}{2}$ revolutions more than the rear wheel?

32. If an emery wheel will stand a surface speed of 2200 ft. per minute, how many revolutions can it make per minute if its diameter is 21 in.? (Use $\pi = 3\frac{1}{7}$.)

33. A boy in a manual-training school is making a bookcase. The distance from the top board to the bottom is 3 ft. $10\frac{1}{2}$ in., inside measure. He wishes to put in three shelves, each $\frac{7}{8}$ in. thick, so that the four book spaces will diminish successively by 1 in. from the bottom to the top. Find the spaces.

Given the following formulas, and using $3\frac{1}{7}$ *for* π *wherever it occurs, find the value of the missing letter:*

Parallelogram

$a = bh$. § 5

	a	b	h
34.		$7\frac{1}{2}$	$9\frac{1}{2}$
35.		$4\frac{3}{4}$	$2\frac{1}{2}$
36.	75	15	
37.	161	23	
38.	319	29	
39.	341		11
40.	403		13
41.	437		19

Circumference

$c = 2\pi r$. § 15

	c	r
48.		7
49.		14
50.		$3\frac{1}{2}$
51.		21
52.	44	
53.	88	
54.	132	
55.	77	

Surface of Sphere

$s = 4\pi r^2$. P. 16

	s	r
62.		3
63.		5
64.		6
65.		10
66.	154	
67.	616	
68.	1386	
69.	2464	

Cylinder

$v = \pi r^2 h$. P. 15

	v	r	h
42.		2	7
43.		7	2
44.		2	14
45.		7	15
46.	4400	10	
47.	1232	14	

Area of Circle

$a = \pi r^2$. § 16

	a	r
56.		2
57.		4
58.		7
59.	154	
60.	616	
61.	1386	

Volume of Sphere

$v = \frac{4}{3}\pi r^3$. P. 16

	v	r
70.		1
71.		2
72.		3
73.	$113\frac{1}{7}$	
74.	$1437\frac{1}{3}$	
75.	$4\frac{4}{21}$	

Given the area of a triangle (§ 6) $a = \frac{1}{2}bh$, *find* b *when:*

76. $a = 15$, $h = 5$.

77. $a = 17.5$, $h = 5$.

From the same formula find h *when:*

78. $a = 8.25$, $b = 5.5$.

79. $a = 10\frac{15}{16}$, $b = 3\frac{1}{8}$.

Given the area of a trapezoid (§ 13) $a = \frac{1}{2}(b + b')h$, *find:*

80. b' when $a = 40$, $b = 6$, and $h = 8$.

81. h when $a = 54$, $b = 8$, and $b' = 10$.

CHAPTER IX

SPECIAL PRODUCTS AND QUOTIENTS

90. Special Products and Quotients. There are certain products that are so often required in algebra that it is helpful to be able to write them at once without the labor of multiplying. In the same way there are certain quotients that we need to be able to write at once without taking the trouble to divide.

91. Square of the Sum or Difference of Two Numbers. If we multiply $a + b$ by $a + b$ the product is $a^2 + 2ab + b^2$, and if we multiply $a - b$ by $a - b$ the product is $a^2 - 2ab + b^2$, as is shown below.

$$\begin{array}{rrr} a + & b & \\ a + & b & \\ \hline a^2 + & ab & \\ & ab & + b^2 \\ \hline a^2 + & 2ab & + b^2 \end{array} \qquad\qquad \begin{array}{rrr} a - & b & \\ a - & b & \\ \hline a^2 - & ab & \\ - & ab & + b^2 \\ \hline a^2 - & 2ab & + b^2 \end{array}$$

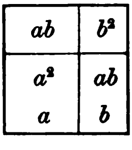
The square of the sum of two numbers is the square of the first, plus twice their product, plus the square of the second.

That is, $\mathbf{(a + b)^2 = a^2 + 2ab + b^2}$.

Therefore

$$13^2 = (10 + 3)^2 = 10^2 + 2 \times 10 \times 3 + 3^2 = 169,$$

$$35^2 = (30 + 5)^2 = 900 + 2 \times 30 \times 5 + 25 = 1225.$$



It is easily seen that the figure representing the square on $a + b$ is made up of a^2, ab, ab, and b^2, or $a^2 + 2ab + b^2$.

The square of the difference of two numbers is the square of the first, minus twice their product, plus the square of the second.

That is, $\mathbf{(a - b)^2 = a^2 - 2ab + b^2}$.

Therefore $37^2 = (40 - 3)^2 = 40^2 - 2 \times 40 \times 3 + 3^2 = 1369$.

Exercise 71. Square of the Sum or Difference

Examples 1 to 46, oral — Examples 47 to 68, written

1. $(x+y)^2$.
2. $(p+q)^2$.
3. $(m+n)^2$.
4. $(a+x)^2$.
5. $(b+y)^2$.
6. $(a+1)^2$.
7. $(a+2)^2$.
8. $(x+3)^2$.
9. $(4+n)^2$.
10. $(5+y)^2$.
11. $(x-y)^2$.
12. $(p-q)^2$.
13. $(m-n)^2$.
14. $(a-x)^2$.
15. $(b-2y)^2$.
16. $(a-2)^2$.
17. $(a-7)^2$.
18. $(x-6)^2$.
19. $(8-n)^2$.
20. $(9-y)^2$.
21. $(2a+b)^2$.
22. $(2a-b)^2$.
23. $(2a+1)^2$.
24. $(2a-1)^2$.
25. $(4x+y)^2$.
26. $(5x-y)^2$.
27. $(x+6y)^2$.
28. $(x-7y)^2$.
29. $(2a+3)^2$.
30. $(2a-3)^2$.
31. 11^2.
32. 12^2.
33. 21^2.
34. 22^2.
35. 16^2.
36. 31^2.
37. 14^2.
38. 41^2.
39. 51^2.
40. 49^2.

41. Divide $x^2+2xy+y^2$ by $x+y$; x^2+2x+1 by $x+1$.

42. Divide $p^2-2pq+q^2$ by $p-q$; p^2-2p+1 by $p-1$.

43. Divide $m^2+2mn+n^2$ by $m+n$; $1+2n+n^2$ by $1+n$.

44. Divide $m^2-2mn+n^2$ by $m-n$; $1-2n+n^2$ by $1-n$.

45. How should you proceed if you wished to write the square of $5x+3y$ without multiplying in the usual way?

46. How should you proceed if you wished to write the square of $9x^3-7y^2$ without multiplying in the usual way?

47. $(2x+3)^2$.
48. $(2x-3)^2$.
49. $(7x+6)^2$.
50. $(7x-6)^2$.
51. $(5x^2+9)^2$.
52. $(5x^2-9)^2$.
53. $(9-5x^2)^2$.
54. $(12x^2y^2+1)^2$.
55. $(12x^2y^2-1)^2$.
56. $(15x^3y^3+2)^2$.
57. $(15x^3y^3-2)^2$.
58. $(2-15x^3y^3)^2$.
59. $(x^4y^3+x^3y^4)^2$.
60. $(x^4y^3-x^3y^4)^2$.
61. $(abc+7)^2$.
62. $(abc-7)^2$.
63. $(7-abc)^2$.
64. $(a^3b^2c+12)^2$.
65. $(a^3b^2c-12)^2$.
66. $(12-a^3b^2c)^2$.
67. $(7a^3b^2c+11)^2$.

68. Square $(a+b)+c$ as if $a+b$ were one term. Then remove the parentheses in the result.

92. Product of the Sum and Difference of Two Numbers. If we multiply $a-b$ by $a+b$, or $a+b$ by $a-b$, the product is a^2-b^2, as is here shown.

$$\begin{array}{lll} a & -b & \\ \underline{a} & \underline{+b} & \\ a^2 & -ab & \\ & \underline{ab} & \underline{-b^2} \\ a^2 & & -b^2 \end{array} \qquad\qquad \begin{array}{lll} a & +b & \\ \underline{a} & \underline{-b} & \\ a^2 & +ab & \\ & \underline{-ab} & \underline{-b^2} \\ a^2 & & -b^2 \end{array}$$

Therefore, *the product of the sum and the difference of two numbers is the difference of their squares.*

That is, $(a+b)(a-b)=a^2-b^2.$

Similarly, $(2x+1)(2x-1)=4x^2-1,$

$$(5a+3b)(5a-3b)=25a^2-9b^2,$$

$$33\times 27=(30+3)(30-3)=900-9=891.$$

Exercise 72. Product of the Sum and Difference

Examples 1 to 23, oral — Examples 24 to 27, written

1. $(x+y)(x-y)$.
2. $(p+q)(p-q)$.
3. $(a+7)(a-7)$.
4. $(9+m)(m-9)$.
5. $(a+6)(6-a)$.
6. $(3x+8)(8-3x)$.
7. $(p^4+3)(p^4-3)$.
8. $(2x^2+1)(2x^2-1)$.
9. $(3x^4+1)(3x^4-1)$.
10. $(5x^7+1)(1-5x^7)$.
11. $(2x+3y)(2x-3y)$.
12. $(3a+9b)(3a-9b)$.
13. $(4a^2+2)(4a^2-2)$.
14. $(a^mb^m+1)(a^mb^m-1)$.
15. What is the product of $20+4$ and $20-4$? of 24 and 16?
16. 32 × 28.
17. 34 × 26.
18. 41 × 39.
19. 42 × 38.
20. 51 × 49.
21. 52 × 48.
22. 61 × 59.
23. 92 × 88.

Multiply as indicated:

24. $(12x^6+16)(12x^6-16)$.
25. $(15a^4x^4+35)(15a^4x^4-35)$.
26. $(36+17b^m)(36-17b^m)$.
27. $(29a^{17}+c^{17})(29a^{17}-c^{17})$.

93. Products of Trinomials. Since we may express the trinomial $a + b + c$ in a binomial form, as $(a + b) + c$ or $a + (b + c)$, we may often apply the principle of § 92 in finding the product of two trinomials.

For example,

$$\begin{aligned}(a + b + c)(a - b - c) &= [a + (b + c)][a - (b + c)]\\ &= a^2 - (b + c)^2\\ &= a^2 - b^2 - 2bc - c^2.\\ (x + y - z)(x - y + z) &= [x + (y - z)][x - (y - z)]\\ &= x^2 - (y - z)^2\\ &= x^2 - y^2 + 2yz - z^2.\end{aligned}$$

Exercise 73. Products of Certain Trinomials

Examples 1 to 9, oral — Examples 10 to 16, written

1. State two ways in which $x + y - z$ may be written in the form of a binomial. Do the same for $x - y + z$.

2. How could $(m + n + 3)(m + n - 3)$ be considered as the product of two binomials? Find this product.

3. Multiply $a + b + 5$ by $a + b - 5$.

Multiply as indicated:

4. $(m + n + 2)(m + n - 2)$.
5. $(p + q + 7)(p + q - 7)$.
6. $(x + y + z)(x + y - z)$.
7. $(a + b + 9)(a + b - 9)$.
8. $(a + b^2 + 1)(a + b^2 - 1)$.
9. $(a + b^2 + 3)(a + b^2 - 3)$.

10. Write $(1 + x - y)$ $(1 - x + y)$ as if it indicated the product of two binomials, and find this product.

Multiply as indicated:

11. $(9 + x + y)(9 + x - y)$.
12. $(a - 7 + b)(a + 7 - b)$.
13. $(x + 2y - z)(x - 2y + z)$.
14. $(x^2 - xy + y^2)(x^2 + xy + y^2)$.
15. $(p^2 + q^2 - pq)(p^2 + q^2 + pq)$.
16. $(a^2 - 2ax + 4x^2)(a^2 + 2ax - 4x^2)$.

94. Products of Binomials. If two binomials have a common term their product can easily be written. For example, study the following:

$$\begin{array}{llll}
x & +7 & & \\
x & +5 & & \\
\hline
x^2 & +7x & & \\
 & 5x & +35 & \\
\hline
x^2 & +12x & +35 &
\end{array}
\qquad\qquad
\begin{array}{llll}
x & +a & & \\
x & +b & & \\
\hline
x^2 & +ax & & \\
 & bx & & +ab \\
\hline
x^2 & +(a+b)x & & +ab
\end{array}$$

The product of two binomials having a common term equals the square of the common term, plus the product of the common term by the sum of the other terms, plus the product of the other terms.

That is, $$(x+a)(x+b)=x^2+(a+b)x+ab.$$

Thus $(x+7)(x-3)=x^2+4x-21,$

because $+7+(-3)=4$, and $+7\times(-3)=-21.$

Similarly, $(x-9)(x-6)=x^2-15x+54,$

$(a+7)(a+7)=a^2+14a+49$ (as in § 91),

and $(a+7)(a-7)=a^2+0-49$ (as in § 92).

Exercise 74. Products of Binomials

Examples 1 to 30, oral — Examples 31 to 68, written

1. $(a+2)(a+3)$.
2. $(a+5)(a+7)$.
3. $(x+9)(x+3)$.
4. $(a+4)(a+5)$.
5. $(a+7)(a+3)$.
6. $(p+9)(p+4)$.
7. $(x+6)(x+2)$.
8. $(x+5)(x+9)$.
9. $(x+2)(x+7)$.
10. $(x+8)(x+7)$.
11. $(a+2)(a-3)$.
12. $(a-5)(a+7)$.
13. $(x+9)(x-3)$.
14. $(a-4)(a+5)$.
15. $(a+7)(a-3)$.
16. $(p-9)(p+4)$.
17. $(x+6)(x-2)$.
18. $(x-5)(x+9)$.
19. $(x+2)(x-7)$.
20. $(x-8)(x+7)$.
21. $(a-2)(a-3)$.
22. $(a-5)(a-7)$.
23. $(x-9)(x-3)$.
24. $(a-4)(a-5)$.
25. $(a-7)(a-3)$.
26. $(p-9)(p-4)$.
27. $(x-6)(x-2)$.
28. $(x-5)(x-9)$.
29. $(x-2)(x-7)$.
30. $(x-8)(x-7)$.

31. Multiply $a-7$ by $a+2$; by $a-2$; by $a+7$; by $a-7$.

32. Multiply $2x+1$ by $2x+3$; by $2x-3$; by $2x-1$.

33. Multiply $7p+4$ by $7p+9$; by $7p-9$; by $7p+4$.

Multiply as indicated:

34. $(3a+7)(3a+19)$.
35. $(3a+7)(3a-19)$.
36. $(3a-7)(3a+19)$.
37. $(3a-7)(3a-19)$.
38. $(3a+7)(19+3a)$.
39. $(3a-7)(19+3a)$.
40. $(4x+5)(4x-9)$.
41 $(7x+3)(7x-11)$.
42 $(8p+q)(8p-4q)$.
43. $(9p+4q)(9p+7q)$.
44. $(5m+3n)(5m-11n)$.
45. $(7m+4n)(7m-9n)$.
46. $(6a+3b)(6a-7b)$.
47. $(7a-3b)(7a+9b)$.
48. $(ab+9)(ab-17)$.
49. $(3ab+7)(3ab+23)$.
50. $(a^2b^2+9)(a^2b^2+8)$.
51. $(a^3b^3+4)(a^3b^3-15)$.
52. $(8+pq)(pq-27)$.
53. $(7-pq)(9-pq)$.
54. $(a^n+4)(a^n-17)$.
55. $(5a^m+16)(5a^m-27)$.
56. $(a^mb^n+4)(a^mb^n+7)$.
57. $(a^mb^n+4)(a^mb^n-15)$.

58. Since $(a+7)(a+9)=a^2+16a+63$, what is the quotient of $a^2+16a+63$ divided by $a+7$? divided by $a+9$?

Divide:

59. $a^2+7a+12$ by $a+3$.
60. $a^2+9a+20$ by $a+4$.
61. $a^2+8a+12$ by $a+2$.
62. p^2-p-12 by $p+3$.
63. $p^2+2p-35$ by $p+7$.
64. $p^2-5p-36$ by $p-9$.
65. $m^2+10m+21$ by $m+7$.
66. $m^2-4m-21$ by $m-7$.

67. If $f(a)=a+15$ and $F(a)=a+17$, find $f(a)\times F(a)$; $(a-3)\times F(a)$; $(a-7)\times f(a)$; $5\times f(a)\times F(a)$.

68. If the side of a square is 4.2 in. what is the area? What would be the increase in area if we took the side as $(4.2+x)$ in.? the decrease, if we took the side as $(4.2-x)$ in.? To the nearest hundredth of a square inch, what is this increase or decrease if $x=0.02$? Also to the nearest tenth?

95. The Cube of a Binomial. If we raise $a+b$ to the third power we shall have $(a+b)(a+b)(a+b)$, which is the same as $(a+b)(a+b)^2$, or $(a+b)(a^2+2ab+b^2)$. Multiplying, we have

$$\begin{array}{llll}
a^2 & +2ab & +b^2 & \\
a & +b & & \\
\hline
a^3 & +2a^2b & +ab^2 & \\
 & a^2b & +2ab^2 & +b^3 \\
\hline
a^3 & +3a^2b & +3ab^2 & +b^3
\end{array}$$

The cube of the sum of two numbers is the cube of the first, plus three times the square of the first multiplied by the second, plus three times the first multiplied by the square of the second, plus the cube of the second.

That is, $\boldsymbol{(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3}$,

and $\boldsymbol{(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3}$.

Similarly, $(a+2)^3 = a^3 + 3a^2\cdot 2 + 3a\cdot 2^2 + 2^3$

$= a^3 + 6a^2 + 12a + 8.$

Exercise 75. The Cube of a Binomial

Examples 1 to 10, oral — Examples 11 to 21, written

1. $(x+y)^3$.
2. $(x-y)^3$.
3. $(p+q)^3$.
4. $(p-q)^3$.
5. $(m+n)^3$.
6. $(a+1)^3$.
7. $(a-1)^3$.
8. $(1-a)^3$.
9. $(1-x)^3$.
10. $(1-y)^3$.
11. $(a^2+1)^3$.
12. $(a^2-1)^3$.
13. $(1+a^2)^3$.
14. $(1-a^2)^3$.
15. $(1-x^2)^3$.
16. Cube $a+3$; $a-3$; $3+a$; $3-a$; a^2+3; a^2-3.
17. Cube $2a+b$; $2a-b$; $a+2b$; $a-2b$; $b-2a$.
18. Cube $3a+2b$; $3a-2b$; $2a+3b$; $2a-3b$; $3b-2a$
19. Cube $4a+1$; $4a-1$; $1-4a$; $a+4$; $4+a$; $4-a$.
20. Cube $3a+7$. Evaluate for $a=10$.
21. Evaluate $4n+1$ and also its cube for $n=10$, and thus find the cube of 41.

96. Quotients from dividing Squares or the Difference of Squares.

Since $a^2 + 2ab + b^2 = (a + b)(a + b)$,

therefore $$\frac{a^2 + 2ab + b^2}{a + b} = a + b.$$

Since $a^2 - 2ab + b^2 = (a - b)(a - b)$,

therefore $$\frac{a^2 - 2ab + b^2}{a - b} = a - b.$$

Since $a^2 - b^2 = (a + b)(a - b)$,

therefore $$\frac{a^2 - b^2}{a + b} = a - b,$$

and $$\frac{a^2 - b^2}{a - b} = a + b.$$

Exercise 76. Special Quotients

Examples 1 to 12, oral — Examples 13 to 21, written

1. Divide $x^2 + 4x + 4$ by $x + 2$.
2. Divide $a^2 - 6a + 9$ by $a - 3$.
3. Divide $25x^2 - 4$ by $5x + 2$; by $5x - 2$.

Divide:

4. $p^2 - q^2$ by $p + q$.
5. $9x^4 - y^4$ by $3x^2 + y^2$.
6. $25a^4 - 1$ by $5a^2 - 1$.
7. $16 - x^4$ by $4 - x^2$.
8. $81 - m^4$ by $9 - m^2$.
9. $49p^4 - 64q^4$ by $7p^2 + 8q^2$.

State two binomials whose product is:

10. $4x^2 - 25y^2$.
11. $36p^2q^2 - 1$.
12. $49a^2b^2c^2 - 1$.
13. $(a + b)^2 - c^2$.
14. $a^2 - (b + c)^2$.
15. $a^2 - (b - c)^2$.

Divide:

16. $x^6 - 1$ by $x^3 + 1$.
17. $4a^6 - 9$ by $2a^3 - 3$.
18. $9p^6 - 25$ by $3p^3 + 5$.
19. $(x + y)^2 - z^2$ by $x + y - z$.
20. $(x - y)^2 - z^2$ by $x - y - z$.
21. $(x - y)^2 - z^2$ by $x - y + z$.

97. Dividing the Sum or Difference of Two Cubes. If we divide $a^3 + b^3$ by $a + b$, we obtain $a^2 - ab + b^2$. Therefore

The sum of the cubes of two numbers is divisible by the sum of the numbers, and the quotient is the square of the first, minus their product, plus the square of the second.

That is,
$$\frac{a^3+b^3}{a+b} = a^2 - ab + b^2.$$

If we divide $a^3 - b^3$ by $a - b$, we obtain $a^2 + ab + b^2$. Therefore

The difference of the cubes of two numbers is divisible by the difference of the numbers, and the quotient is the square of the first, plus their product, plus the square of the second.

That is,
$$\frac{a^3-b^3}{a-b} = a^2 + ab + b^2.$$

For example, $(m^3n^3 - p^6) \div (mn - p^2) = m^2n^2 + mnp^2 + p^4$.

Exercise 77. Dividing the Sum or Difference of Cubes

Examples 1 to 9, oral — Examples 10 to 22, written

1. $(x^3 + y^3) \div (x + y)$.
2. $(x^3 - y^3) \div (x - y)$.
3. $(p^3 + q^3) \div (p + q)$.
4. $(p^3 - q^3) \div (p - q)$.
5. $(m^3 + 1) \div (m + 1)$.
6. $(m^3 - 1) \div (m - 1)$.
7. $(1 - m^3) \div (1 - m)$.
8. $(x^3 + 8) \div (x + 2)$.
9. $(x^3 - 8) \div (x - 2)$.
10. $(a^3b^3 + c^3) \div (ab + c)$.
11. $(a^3b^3 - c^3) \div (ab - c)$.
12. $(c^3 - a^3b^3) \div (c - ab)$.
13. $(8a^3 + b^3) \div (2a + b)$.
14. $(8a^3 - b^3) \div (2a - b)$.
15. $(b^3 - 8a^3) \div (b - 2a)$.
16. $(27x^3 + 1) \div (3x + 1)$

Divide as indicated:

17. $\dfrac{125a^3 + 8b^3}{5a + 2b}$.
18. $\dfrac{64a^3b^3 + 8c^3}{4ab + 2c}$.
19. $\dfrac{343m^3 + n^3}{7m + n}$.
20. $\dfrac{343m^3 - n^3}{7m - n}$.
21. $\dfrac{8x^3 - 125y^3}{2x - 5y}$.
22. $\dfrac{8x^3 - 64y^3z^3}{2x - 4yz}$.

98. Summary of Special Products and Quotients. The following special relations mentioned in this chapter should be memorized, both as formulas and in complete sentences:

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$
$$(a+b)(a-b) = a^2 - b^2$$
$$(x+a)(x+b) = x^2 + (a+b)x + ab$$
$$\frac{a^2 + 2ab + b^2}{a+b} = a + b$$
$$\frac{a^2 - 2ab + b^2}{a-b} = a - b$$
$$\frac{a^2 - b^2}{a+b} = a - b, \text{ and } \frac{a^2 - b^2}{a-b} = a + b$$
$$\frac{a^3 + b^3}{a+b} = a^2 - ab + b^2$$
$$\frac{a^3 - b^3}{a-b} = a^2 + ab + b^2$$

Exercise 78. Review of Special Products and Quotients

Examples 1 to 12, oral — Examples 13 to 60, written

1. $(a+m)^2$.
2. $(a-m)^2$.
3. $(a+m)^3$.
4. $(a-m)^3$.
5. $(2a+1)^2$.
6. $(2a-1)^2$.
7. $(1+2a)^2$.
8. $(1-2a)^2$.
9. $(p+q)^3$.
10. $(p-q)^3$.
11. $(q-p)^3$.
12. $(q-1)^3$.
13. Multiply $x+7$ by $x+7$; by $x-7$; by $x+8$; by $x-8$.

Divide as indicated:

14. $\dfrac{4m^2 + 4m + 1}{2m+1}$.
15. $\dfrac{4m^2 - 4m + 1}{2m-1}$.
16. $\dfrac{49a^2b^2 + 14ab + 1}{7ab+1}$.
17. $\dfrac{49a^2b^2 - 14ab + 1}{7ab-1}$.

Divide as indicated:

18. $\dfrac{0.04\,m^2 - 1}{0.2\,m + 1}$. **19.** $\dfrac{49\,a^2b^2 - 1}{7\,ab - 1}$. **20.** $\dfrac{121\,m^2 - 25}{11\,m + 5}$.

21. Multiply $x^2 + y^2 + xy$ by $x^2 + y^2 - xy$.

22. Multiply $a^2 - 2\,ab + b^2$ by $a^2 + 2\,ab + b^2$.

23. If $f(x) = x^2 + 2\,x + 1$, and $F(x) = x^2 - 2\,x + 1$, find $f(x) \cdot F(x)$; the square of $f(x) - 1$; the square of $F(x) - 1$.

24. If $f(a) = a + 1$, and $F(a) = a - 1$, find the square of $f(a)$; the square of $F(a)$; the product of these squares.

25. If $f(a) = a^5 + 32$, and $F(a) = a^5 - 32$, find the square of $f(a)$; the square of $F(a)$; the product of these squares.

26. If $f(m) = m^2 - m + 5$, and $F(m) = m^2 + m + 5$, find $f(m) \cdot F(m)$; the square of $f(m)$; the square of $F(m)$.

27. Divide $(x + y)^2 + 2\,(x + y) + 1$ by $x + y + 1$.

28. Divide $(x - y)^2 + 2\,(x - y) + 1$ by $x - y + 1$.

29. Divide $(x + y)^2 - (x - y)^2$ by $(x + y) - (x - y)$, without simplifying either expression. Then simplify each and divide.

30. Divide $(x + y)^3 + (x - y)^3$ by $(x + y) + (x - y)$, without simplifying either expression. Then simplify each and divide.

31. Divide $(x + y)^3 - (x - y)^3$ by $(x + y) - (x - y)$, without simplifying either expression. Then simplify each and divide.

Divide as indicated:

32. $\dfrac{x^2 + 4\,xy + 4\,y^2 - 1}{x + 2\,y - 1}$. **35.** $\dfrac{36\,x^2 + 48\,xy + 16\,y^2 - 49}{6\,x + 4\,y + 7}$.

33. $\dfrac{9\,a^2 + 12\,ab + 4\,b^2 - 25}{3\,a + 2\,b + 5}$. **36.** $\dfrac{49\,a^2 + 56\,ab + 16\,b^2 - 64}{7\,a + 4\,b - 8}$.

34. $\dfrac{16p^2 - 24pq + 9q^2 - 36}{4\,p - 3\,q - 6}$. **37.** $\dfrac{81\,x^2 - 90\,xy + 25\,y^2 - 121}{9\,x - 5\,y + 11}$.

38. $\dfrac{a^3 + 3\,a^2b + 3\,ab^2 + b^3 + x^3 + 3\,x^2y + 3\,xy^2 + y^3}{a + b + x + y}$.

39. $\dfrac{a^3 - 3\,a^2b + 3\,ab^2 - b^3 - x^3 + 3\,x^2y - 3\,xy^2 + y^3}{a - b - x + y}$.

Multiply as indicated:

40. $(2x - 3y - z)(2x - 3y + z)$.

41. $(7x - 5y + 3z)(5y + 3z + 7x)$.

42. $(3x^2 - 7 + 2y^3)(7 + 3x^2 + 2y^3)$.

43. $(7ab - 4 - 7c^2)(7c^2 + 4 + 7ab)$.

44. $(5x + 3y - \frac{1}{2})(5x - 3y + \frac{1}{2})$.

45. $(9x^2 - 7xy + y^2)(y^2 + 9x^2 + 7xy)$.

46. $(a^2 - 5b^2 - 12c^2)(a^2 - 5b^2 + 12c^2)$.

47. $(15x^2 - 13y^2 + 3z^2)(15x^2 + 13y^2 + 3z^2)$.

48. Divide the sum of the cubes of $17x$ and $19y$ by $17x + 19y$.

49. Divide the cube of $12a$ minus the cube of $7b$ by $12a$ minus $7b$; also the sum of the cubes by the sum of the monomials.

50. Divide $x^2 + 16x + 63$ by $x + 7$; also by $x + 9$.

51. Divide $x^2 - 3x - 54$ by $x + 6$; also by $9 - x$.

52. Divide $x^2 - (y - z)^2$ by $x + y - z$.

53. Divide $x^2 - x - 132$ by $x - 12$. What other exact divisor has it, and what is the quotient corresponding to that divisor?

54. Multiply $x + 13$ by $x + 13$; by $x - 13$; by $x + 11$.

55. Multiply $2x + 21$ by $2x + 21$; by $2x - 21$; by $21 - 2x$.

56. Multiply $a + x + y$ by $a - x - y$; by $a - x + y$; by $a + x - y$; by $-a - x - y$.

57. Multiply $a^2 + 2ab + b^2$ by $a + b$; by $(a + b)^2$; by $a^2 - 2ab + b^2$.

58. Multiply $4a^2 - 12ab + 9b^2$ by $2a - 3b$, and $4a^2 + 12ab + 9b^2$ by $2a + 3b$.

59. Multiply $x + 13$ by $x + 7$; by $x + 11$; by $x + 13$; by $x + 21$; by $x + 17$; by $x - 13$; by $x - 17$.

60. Multiply $2x + 11$ by $2x - 11$; by $2x + 11$; by $2x + 13$; by $2x - 13$; by $2x + 15$; by $2x - 15$.

CHAPTER X

FACTORS

99. Factor. Any one of two or more numbers which multiplied together form a product is called a *factor* of the product.

As already stated (§ 38), 2 and 3 are factors of 6; 2, π, and r are factors of $2\pi r$, and $(b + b')$ and h are factors of $(b + b')h$.

A factor containing a letter is called a *literal factor*; one that is expressed by a numeral is called a *numerical factor*. In the expression $2a(4 + c)$, 2 is a numerical factor; a and $4 + c$ are literal factors.

Except where the contrary is stated, factors are limited to expressions that do not contain fractions or indicated roots. For example, although $3 \cdot \frac{2}{3} = 2$, and $n \cdot \frac{a}{n} = a$, we do not speak of 3 and $\frac{2}{3}$ as factors of 2, nor of n and $\frac{a}{n}$ as factors of a. In the same way, although $(\sqrt{x} + \sqrt{a})(\sqrt{x} - \sqrt{a}) = x - a$ (§ 92), we would not, at this stage, speak of $x - a$ as factorable. If fractions occur as coefficients they are admitted as factors. For example, $\frac{2}{3}a + \frac{2}{3}b = \frac{2}{3}(a + b)$.

100. Prime and Factorable Expressions. An expression that contains no factors except itself and unity is said to be *prime*.

An expression that contains factors other than itself and unity is said to be *factorable*.

In factoring an expression we separate it into its prime factors. Thus the factors of a^3 are given as a, a, and a, although a^2 is also a factor of a^3.

101. Reduction of Fractions. We use factoring in reducing fractions to lowest terms, exactly as in arithmetic.

Just as $\frac{6}{15}$ is reduced to $\frac{2}{5}$ by canceling the common factor 3, that is, by dividing both terms by 3, so $\frac{a^2}{ab}$ is reduced to $\frac{a}{b}$ by canceling the common factor a. The subject is more fully treated in Chapter XI.

102. Factors of Monomials. The factors of a monomial are found by inspection.

For example, the factors of $6a^2b^3$ are 6, a, a, b, b, b. In factoring a monomial we do not attempt to factor the numerical coefficient. Similarly, we speak of the factors of $6a + 6b$ as 6 and $a + b$, although the 6 is also factorable.

To reduce $\frac{4}{6}$ to lowest terms we divide both terms by 2, the fraction becoming $\frac{2}{3}$. This is called canceling the factor 2. Similarly, to reduce $\frac{a^2b}{ab^2}$ to lowest terms we divide both terms by (or cancel) the common factors a and b, that is, the factors that are in both a^2b and ab^2, the result being $\frac{a}{b}$.

Exercise 79. Factors of Monomials

Examples 1 to 9, oral — Examples 10 to 27, written

1. Find the factors of a^2b^3; of a^3b^2; of $3a^2x^2y^2$; of $5ax^3y^4$.

2. Find the common factors of a^3b^2 and a^2b^3; of x^4y^4 and x^5y^3.

3. Find the common factors of m^7n^2 and m^5n^6; of x^9y^3 and x^3y^9; of abc^2 and a^2bc; of ab^2cd and abc^2d.

Reduce to lowest terms by canceling the factors common to numerator and denominator :

4. $\frac{a}{ab}$.
5. $\frac{b}{ab}$.
6. $\frac{a^2}{ab}$.
7. $\frac{b^2}{ab}$.
8. $\frac{ab}{a^2b^2}$.
9. $\frac{a^2b}{a^3b^2}$.
10. $\frac{a^4b^3}{a^3b^4}$.
11. $\frac{a^5b^7}{a^9b^4}$.
12. $\frac{a^9b^6}{a^5b^3}$.
13. $\frac{x^3y^3}{x^2y^2}$.
14. $\frac{x^7y^5}{x^2y^3}$.
15. $\frac{x^9y^3}{x^4y^3}$.
16. $\frac{3a^2b}{9ab^2}$.
17. $\frac{6a^4b^2}{9a^2b^2}$.
18. $\frac{8x^7y^3}{12x^7y^4}$.
19. $\frac{8m^6n^6}{14m^2n^2}$
20. $\frac{5m^9n^3}{15m^3n^9}$.
21. $\frac{8p^7q^3}{20pq}$.
22. $\frac{17p^7q^3}{34pqr}$.
23. $\frac{17a^2b^2c^2}{51abc}$.
24. $\frac{19abc}{57a^2b^2c^2}$.
25. $\frac{17a^3b^2c}{85ab^2c^3}$.
26. $\frac{21p^4q^2r^4}{49p^2q^4r^2}$.
27. $\frac{27x^7y^7z^7}{81xyz}$.

103. Monomial Factors of Polynomials. When the terms of a polynomial have a common monomial factor the expression may be factored by inspection.

Required to factor $4a^3 + 6a^2b$.

It is evident that 2 is a factor of both terms, and that a^2 is also a factor of both. We may therefore write the expression thus:

$$4a^3 + 6a^2b = 2a^2 \cdot 2a + 2a^2 \cdot 3b = 2a^2(2a + 3b).$$

While the factors of a^2 are a and a, it is customary to leave the result in the form given above.

We may check the work by assigning numerical values to the letters, as in multiplication and division. In the above case, if $a = 1$ and $b = 2$ we have $4 + 12 = 2(2 + 6)$.

Exercise 80. Monomial Factors of Polynomials

Examples 1 to 10, oral — Examples 11 to 22, written

1. Factor $ax + bx$; $ax + ay$; $ax + bx + cx$; $ax + ay + az$.
2. Factor $ax^2 - bx^2$; $a^2x - a^2y$; $a^2x + b^2x + c^2x + d^2x$.

Factor the following:

3. $5x^2 + 15x^3$.
4. $8x^2 - 12x^4$.
5. $6a^3 + 9a^2$.
6. $9a^2 - 6a^5$.
7. $a^3 + a^2 + a$.
8. $a^3 + 3a^2 - 4a$.
9. $x^4 + x^3y - x^2y^2$.
10. $a^5 - 3a^4 + 7a^3$.
11. $-x^4 - 3x^3 - 2x^2$.
12. $6a^2b - 9ab^2 + 3ab$.
13. $4p^3q + 6pq^3 - 8p^2q^2$.
14. $8x^4y^4 - 6x^2y^2 + 10x^3y^3$.
15. $5m^7n^2 - 10m^5n^3 + 15m^4n^4$.
16. $12abc + 8a^2b^2c^2 - 4a^3b^3c^3$.
17. $6a^5b^4c^3 - 4a^4b^3c^2 + 2a^3b^2c$.
18. $3x^3y^3z^3 + 6x^2y^2z^2 + 9xyz$.
19. $17x^6y^6 - 51x^4y^4 + 85x^2y^2$.
20. $-19m^2n^2 + 76m^3n^3 - 95m^4n^4$.

Factor the numerator and denominator. Then reduce to lowest terms by canceling all common factors from each:

21. $\dfrac{12a^3b^3c^3 - 15a^2b^2c^2}{15a^2b^2c^2 - 12abc}$.
22. $\dfrac{5x^2y^2 - 10x^3y^3 + 15x^4y^4}{5x^3y^3 - 5x^2y^2 - 10x^4y^4}$.

104. Polynomials factored by grouping Terms. If we do not find a monomial factor that is common to all the terms of a polynomial, we may frequently factor by grouping the terms as shown in the following examples:

1. Required to factor $ax + ay + bx + by$.

Factoring the first two terms and the last two terms, we have

$$ax + ay + bx + by = a(x + y) + b(x + y).$$

We now see that $x + y$ is a common factor of these two groups. Hence

$$a(x + y) + b(x + y) = (a + b)(x + y).$$

We may check the work by multiplying $x + y$ by $a + b$, the result being $ax + ay + bx + by$; or we may substitute 1 (or any other number) for the letters. If we substitute 1 for each letter in the factors, we have

$$(1 + 1)(1 + 1) = 2 \cdot 2 = 4,$$

and $$ax + ay + bx + by = 1 + 1 + 1 + 1 = 4.$$

2. Required to factor $3ax + 4ay + 3bx + 4by$.

$$\begin{aligned} 3ax + 4ay + 3bx + 4by &= a(3x + 4y) + b(3x + 4y) \\ &= (a + b)(3x + 4y). \end{aligned}$$

3. Required to factor $ax + ay - bx - by$.

$$\begin{aligned} ax + ay - bx - by &= (ax + ay) - (bx + by) \\ &= a(x + y) - b(x + y) \\ &= (a - b)(x + y). \end{aligned}$$

4. Required to factor $a^2y + ab^2 - axy - b^2x$.

$$\begin{aligned} a^2y + ab^2 - axy - b^2x &= (a^2y + ab^2) - (axy + b^2x) \\ &= a(ay + b^2) - x(ay + b^2) \\ &= (a - x)(ay + b^2). \end{aligned}$$

5. Required to factor $a^2x + b^2x + a^2 + b^2$.

$$\begin{aligned} a^2x + b^2x + a^2 + b^2 &= x(a^2 + b^2) + (a^2 + b^2) \\ &= (x + 1)(a^2 + b^2). \end{aligned}$$

6. Required to factor $m^2x + m^2y + m^2z + x + y + z$.

$$\begin{aligned} m^2x + m^2y + m^2z + x + y + z &= m^2(x + y + z) + (x + y + z) \\ &= (m^2 + 1)(x + y + z). \end{aligned}$$

In all the above cases we may check by actual multiplication or by substituting 1 (or any other number) for the letters.

Exercise 81. Polynomials factored by grouping Terms

Examples 1 to 8, oral — Examples 9 to 34, written

1. Factor $x^4 + x^3$; $xy + y$; $ax + ay^2$; $ax + b^2x$.

2. Factor $p^2 + pq$; $pq + q^2$; $pq + p^2q^2$; $p^2q^2 - pq$.

Name a factor common to the first two terms, and another factor common to the last two terms, of the following:

3. $a^5 + 2a^2 + a^3b + 2b$.
4. $a^2b + ab^2 + 3a + 3b$.
5. $x^2y + xy^2 + xy + y^2$.
6. $m^3n^3 + m^2n^2 + 2mn + 2$.
7. $m^5n^2 - m^2n^5 + 7m^3 - 7n^3$.
8. $pqr + qrx + p^2 + px$.

Factor Exs. 3–8, and also the following:

9. $ax^2 + bx^2 + ay^2 + by^2$.
10. $ax^2 - bx^2 + ay^2 - by^2$.
11. $ax^2 + bx^2 - ay^2 - by^2$.
12. $ax^2 + ay^2 - bx^2 - by^2$.
13. $a^3 + a^2b + 3a + 3b$.
14. $a^4 + a^3b^3 + ab + b^4$.
15. $x^7 + x^4y + x^3y + y^2$.
16. $x^4 - x^3y + xy^2 - y^3$.
17. $p^3 + p^2q + pq^2 + q^3$.
18. $p^5 + pq^4 - p^4q - q^5$.
19. $6ab + 9a + 4b + 6$.
20. $a^3 + a^2b + a + b$.
21. $x^2 + ax + x + a$.
22. $x^2 - ax + x - a$.
23. $x^2y^2 + xy + xyz + z$.
24. $a^2 + am + an + mn$.
25. $a^3 + a^2b^2 + ab + b^3$.
26. $a^3b^3 + ab + a^2b^2 + 1$.

Factor the numerator and then reduce to lowest terms:

27. $\dfrac{a^2 + ab + 2a + 2b}{a + 2}$.

28. $\dfrac{m^2 + mn + m + n}{m + 1}$.

29. $\dfrac{a^2 + 2ab + a + 2b}{a + 1}$.

30. $\dfrac{ax^2 + bx^2 + ay + by}{x^2 + y}$.

31. $\dfrac{ab^2 + b^2 + 2a + 2}{b^2 + 2}$.

32. $\dfrac{a^3b^3 + a^2b^2 + ab + 1}{ab + 1}$.

33. $\dfrac{a^3m^3 - a^2m^2 + am - 1}{am - 1}$.

34. $\dfrac{a^2b^2c^2 - a^3b^3c^3 + 1 - abc}{1 - abc}$.

105. Trinomials that are Perfect Squares. We have learned (§ 91) that a trinomial is a perfect square if it is of the form $a^2 + 2ab + b^2$ or the form $a^2 - 2ab + b^2$.

For $a^2 + 2ab + b^2 = (a + b)^2$, and $a^2 - 2ab + b^2 = (a - b)^2$.

A common form of expression is $a^2 \pm 2ab + b^2$, which is read "a^2 plus or minus 2 ab plus b^2." This is a convenient way of combining two expressions like $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$ in one.

Therefore *a trinomial, arranged according to the descending powers of one of its letters, is a perfect square if the first and last terms are positive and are perfect squares and the middle term is plus or minus twice the product of the square roots of the first and last terms.*

Thus $a^2 + 2ab$ will become a perfect square if b^2 is added. It then becomes $a^2 + 2ab + b^2$. Similarly, to make $-2ab + b^2$ a perfect square add a^2. It then becomes $a^2 - 2ab + b^2$.

To make $16x^2 + (\quad) + 9$ a perfect square, replace the parentheses by $2 \cdot 4x \cdot 3$, or $24x$. It then becomes $16x^2 + 24x + 9$, or $(4x + 3)^2$.

Exercise 82. Trinomials that are Perfect Squares

Examples 1 to 10, oral — Examples 11 to 21, written

1. Add a term to $x^2 - 2xy$ that will make it a perfect square.

In the following replace the parentheses by a term that will make the trinomial a perfect square:

2. $x^2 + 2x + (\quad)$.
3. $(\quad) + 2mn + n^2$.
4. $p^2 + (\quad) + q^2$.
5. $a^2 - 2ax + (\quad)$.
6. $p^2 - (\quad) + x^2$.
7. $(\quad) - 2m^2n^2 + n^4$.
8. $x^4 + 2x^2 + (\quad)$.
9. $m^4 - (\quad) + 1$.
10. $(\quad) - 2xy + 1$.
11. $16x^2 + 24x + (\quad)$.
12. $25x^2 - (\quad) + 4$.
13. $(\quad) - 8xy + y^2$.
14. $9x^2 - 12xy + (\quad)$.
15. $9x^2 - (\quad) + 16y^2$.
16. $81x^2 + 18xy + (\quad)$.
17. $(\quad) - 36xy + 4y^2$.
18. $81x^2 + (\quad) + 9y^2$.
19. $(\quad) - 54xy + 9y^2$.
20. $121a^2b^2 + (\quad) + 1$.
21. $144x^2y^2 + (\quad) + 1$.

106. Factoring Trinomials that are Perfect Squares. It is evident from § 105 that we can easily factor a trinomial if it is a perfect square.

Extract the square root of the first and last terms, and connect these square roots by the sign of the middle term.

That is, $\mathbf{a^2 + 2ab + b^2 = (a + b)(a + b)}$,
and $\mathbf{a^2 - 2ab + b^2 = (a - b)(a - b)}$.

1. Factor $x^2 + 2x + 1$.

Since the first and last terms are respectively the squares of x and 1, and $2x$ is twice the product of x and 1, we have

$$x^2 + 2x + 1 = (x + 1)(x + 1).$$

2. Factor $x^6 - 2x^3y + y^2$.

$$x^6 - 2x^3y + y^2 = (x^3 - y)(x^3 - y).$$

Exercise 83. Factoring Trinomials that are Perfect Squares

Examples 1 to 11, oral — Examples 12 to 21, written

1. Find the square root of $a^2 + 2a + 1$; of $a^2 - 2a + 1$.
2. Find two equal factors of $a^4 + 2a^2 + 1$; of $a^2b^2 + 2ab + 1$.

Factor the following orally:

3. $b^2 + 2bc + c^2$.
4. $b^2 - 2bc + c^2$.
5. $c^2 - 2cd + d^2$.
6. $d^2 + 2d + 1$.
7. $d^2 - 2d + 1$.
8. $d^2 + 4d + 4$.
9. $x^2 - 4x + 4$.
10. $x^2 - 6x + 9$.
11. $x^2 - 8x + 16$.

Write the factors of the following:

12. $49a^2 + 14ab + b^2$.
13. $49a^2 - 14ab + b^2$.
14. $81a^2 + 36ab + 4b^2$.
15. $81a^2 - 54ab + 9b^2$.
16. $121x^2 + 22xy + y^2$.
17. $121x^2 - 44xy + 4y^2$.
18. $121x^2 + 66xy + 9y^2$.
19. $121x^2 - 88xy + 16y^2$.
20. $49x^4y^4 + 42x^2y^2 + 9$.
21. $49x^4y^4 - 28x^2y^2 + 4$.

107. Factoring the Difference of Two Squares. We have found (§ 92) that the difference of the squares of two quantities is always factorable.

The difference of the squares of two quantities is the product of the sum and difference of the quantities.

That is, $a^2 - b^2 = (a + b)(a - b)$.

Factor the binomial $25x^2 - 36y^2$.

The sum of the square roots of $25x^2$ and $36y^2$ is $5x + 6y$, and their difference is $5x - 6y$.

Therefore $25x^2 - 36y^2 = (5x + 6y)(5x - 6y)$.

Exercise 84. Factoring the Difference of Two Squares

Examples 1 to 25, oral — Examples 26 to 36, written

1. Factor $a^2 - b^2$; $x^2 - y^2$; $m^2 - n^2$; $p^2 - q^2$; $a^2 - 1$; $m^2 - 1$; $p^2 - 4$; $x^2 - 7^2$; $8^2 - x^2$; $8^2 - 7^2$.

2. Factor $a^2 - 9$; $b^2 - 16$; $c^2 - 25$; $d^2 - 36$; $p^2 - 49$; $q^2 - 64$; $64 - q^2$; $64 - a^2b^2$; $17^2 - x^2$; $x^2 - 13^2$; $17^2 - 13^2$.

3. Factor $25 - a^2$; $36 - b^2$; $49 - c^2$; $64 - d^2$; $81 - x^2$; $100 - y^2$; $100 - x^2y^2$; $x^2y^2 - 100$; $25^2 - p^2$; $q^2 - 15^2$; $25^2 - 15^2$.

Factor the following:

4. $a^2 - 4$.
5. $a^2 - 4b^2$.
6. $4a^2 - b^2$.
7. $a^2 - 9b^2$.
8. $9a^2 - b^2$.
9. $16a^2 - b^2$.
10. $b^2 - 16a^2$.
11. $25a^2 - b^2$.
12. $b^2 - 25a^2$.
13. $9x^2 - 16y^2$.
14. $16x^2 - 9y^2$.
15. $9x^2 - 25y^2$.
16. $25x^2 - 9y^2$.
17. $36x^2 - 25y^2$.
18. $36x^2 - 9y^2$.
19. $49x^2 - 36y^2$.
20. $64x^2 - 49y^2$.
21. $81x^2 - 64y^2$.
22. $121x^2 - 1$.
23. $121x^4 - 4$.
24. $121x^4 - 9$.
25. $9x^4 - 121$.
26. $144x^2 - 0.25$.
27. $169x^2 - 36$.
28. $169x^4 - 0.09$.
29. $144x^4 - 0.25$.
30. $400x^2 - 9$.
31. $1.44x^2 - 1.21$.
32. $\frac{9}{4}x^2y^2 - \frac{4}{9}z^2$.
33. $2\frac{1}{4}m^2 - 6\frac{1}{4}n^2$.
34. $1\frac{7}{9}a^2 - 2\frac{1}{4}b^2$.
35. $1.69p^2 - 0.81q^2$.
36. $1.21x^2 - 0.16y^2$.

108. Difference of Two Squares. Special Case. If one of the factors is itself the difference of two squares, it can also be factored. For example, required to factor $a^4 - b^4$.

$$\begin{aligned} a^4 - b^4 &= (a^2 + b^2)(a^2 - b^2) \\ &= (a^2 + b^2)(a + b)(a - b). \end{aligned}$$

Similarly, required to factor $x^8 - y^8$.

$$\begin{aligned} x^8 - y^8 &= (x^4 + y^4)(x^4 - y^4) \\ &= (x^4 + y^4)(x^2 + y^2)(x^2 - y^2) \\ &= (x^4 + y^4)(x^2 + y^2)(x + y)(x - y). \end{aligned}$$

Exercise 85. Factoring the Difference of Two Squares

Examples 1 to 9, oral — Examples 10 to 31, written

1. Factor $a^4 - 1$ into two factors. Then factor again.

Separate the following into prime factors:

2. $x^4 - y^4$.
3. $x^4 - 1$.
4. $1 - x^4$.
5. $x^4y^4 - 1$.
6. $1 - x^4y^4$.
7. $x^4 - 16$.
8. $16 - x^4$.
9. $x^4 - 16y^4$.
10. $0.0016x^4 - 1$.
11. $1 - 0.0016x^4$.
12. $0.0016x^4 - y^4$.
13. $0.0081x^4 - 1$.
14. $81x^4 - 0.0016$.
15. $81x^4 - 16y^4$.
16. $16x^4 - 0.0081$.
17. $16x^4 - 0.0081y^4$.
18. $x^8 - y^8$.
19. $x^8 - 1$.
20. $x^8y^8 - 1$.
21. $x^8y^8 - z^8$.
22. $x^8y^8z^8 - 1$.
23. $1 - x^8y^8z^8$.
24. $x^8y^8 - a^8b^8$.
25. $x^8y^8z^8 - a^8$.

Factor the numerator and denominator. Then reduce the fraction to lowest terms by canceling common factors from numerator and denominator.

26. $\dfrac{a^2 - 1}{a^4 - 1}$.
27. $\dfrac{a + 1}{a^4 - 1}$.
28. $\dfrac{a - 1}{a^4 - 1}$.
29. $\dfrac{1 - x^4}{1 + x^2}$.
30 $\dfrac{x + y}{x^4 - y^4}$.
31. $\dfrac{x - y}{x^4 - y^4}$.

109. Difference of Two Squares. Special Case. One of the terms in § 107 may be the square of a polynomial.

1. Factor $(a+2b)^2-9c^2$.

Taking the square roots of $(a+2b)^2$ and $9c^2$, and proceeding as on page 130, we have

$$(a+2b)^2-9c^2=(a+2b+3c)(a+2b-3c).$$

2. Factor $x^4-(a-b)^2$.

$$\begin{aligned}x^4-(a-b)^2&=[x^2+(a-b)][x^2-(a-b)]\\&=(x^2+a-b)(x^2-a+b).\end{aligned}$$

Similarly, both of the terms may be squares of polynomials.

Thus, $$\begin{aligned}(a+b)^2-(c-d)^2&=[(a+b)+(c-d)][(a+b)-(c-d)]\\&=(a+b+c-d)(a+b-c+d).\end{aligned}$$

Exercise 86. Factoring the Difference of Two Squares

Examples 1 to 7, oral — Examples 8 to 19, written

1. Factor a^2-x^2; $(a+b)^2-x^2$; $(a-b)^2-x^2$.
2. Factor $(a+b+c)^2-x^2$; $(a+b-c)^2-x^2$; $(a-b+c)^2-x^2$.
3. Factor $(2a+b)^2-c^2$; $(2a-b)^2-c^2$; $(a+2b)^2-c^2$.

Factor the following:

4. $(x+y)^2-z^2$.
5. $(x-y)^2-z^2$.
6. $(m+n)^2-x^2$.
7. $(m-n)^2-x^2$.
8. $a^2-(b+c)^2$.
9. $a^2-(b-c)^2$.
10. $4a^2-(b+c)^2$.
11. $(a-b)^2-(c+d)^2$.
12. $(a-b)^2-(c-d)^2$.
13. $(a+3b)^2-(c+3d)^2$.
14. $(3x-y)^2-(3a-b)^2$.
15. $(a+b)^2-(c+d+e)^2$.
16. $(x-y)^2-(2m+3n-p)^2$.
17. $(x-7)^2-(a+b+c)^2$.

Reduce to lowest terms as on page 131:

18. $\dfrac{a^2-(b+c)^2}{(a+b+c)^2}$.

19. $\dfrac{(a+b)^2-(c+d)^2}{a+b+c+d}$.

110. Difference of Two Squares. Special Case. By properly grouping the terms, a polynomial may often be factored as the difference of two squares.

1. Factor $x^2 + 2xy + y^2 - 16z^2$.

$$\begin{aligned} x^2 + 2xy + y^2 - 16z^2 &= (x + y)^2 - 16z^2 \\ &= (x + y + 4z)(x + y - 4z). \end{aligned}$$

2. Factor $9x^2 - 9z^2 - y^2 - 6yz$.

We see that the first term is the square of $3x$, and that the last three terms contain y and z. Grouping and changing the signs of the terms placed in parentheses preceded by $-$, we have

$$\begin{aligned} 9x^2 - (9z^2 + 6yz + y^2) &= 9x^2 - (3z + y)^2 \\ &= [3x + (3z + y)][3x - (3z + y)] \\ &= (3x + 3z + y)(3x - 3z - y). \end{aligned}$$

3. Factor $a^2 + b^2 - x^2 - y^2 - 2ab + 2xy$.

Grouping the terms containing a and b, and then those containing x and y, we have

$$\begin{aligned} (a^2 - 2ab + b^2) - (x^2 - 2xy + y^2) &= (a - b)^2 - (x - y)^2 \\ &= [(a - b) + (x - y)][(a - b) - (x - y)] \\ &= (a - b + x - y)(a - b - x + y). \end{aligned}$$

Exercise 87. Factoring the Difference of Two Squares

Examples 1 to 3, oral — Examples 4 to 11, written

1. Find the square of a binomial by selecting a group of three terms from $a^2 - x^2 + b^2 - 2ab$.

2. Find the same (see Ex. 1) in $a^2 + 4b^2 - y^2 + 4ab$.

3. Find the same (see Ex. 1) in $4a^2 - x^2 + 9b^2 + 12ab$.

Factor the following:

4. $a^2 - 2ab - c^2 + b^2$.

5. $a^2 - c^2 + b^2 + 2ab$.

6. $4x^2 - 16y^2 + 1 + 4x$.

7. $1 - x^2 + 2xy - y^2$.

8. $x^2 - a^2 - b^2 - 2ab$.

9. $a^2 + 6ab - 4c^2 + 9b^2$.

10. $4x^2 - 12xz + 12yw + 9z^2 - 4w^2 - 9y^2$.

11. $p^4 - q^2 - 6p^2 - 4m^4 + 9 + 4qm^2$.

111. Difference of Two Squares. Special Case. A case of factoring, occasionally required in courses of study, and merely a special case of the difference of two squares, is illustrated in the following examples:

1. Factor $x^4 + x^2y^2 + y^4$.

If the middle term were $2x^2y^2$ we should have the square of $x^2 + y^2$. Therefore if we add x^2y^2 and also subtract it so as not to change the value, we shall have the difference of two squares, thus:

$$\begin{aligned} x^4 + x^2y^2 + y^4 &= x^4 + 2x^2y^2 + y^4 - x^2y^2 \\ &= (x^2 + y^2)^2 - x^2y^2 \\ &= (x^2 + y^2 + xy)(x^2 + y^2 - xy). \\ \therefore x^4 + x^2y^2 + y^4 &= (x^2 + xy + y^2)(x^2 - xy + y^2). \end{aligned}$$

2. Factor $x^4 + 4x^2y^2 + 16y^4$.

The middle term should be $8x^2y^2$ to make the trinomial a perfect square. Therefore

$$\begin{aligned} x^4 + 4x^2y^2 + 16y^4 &= x^4 + 8x^2y^2 + 16y^4 - 4x^2y^2 \\ &= (x^2 + 4y^2)^2 - 4x^2y^2 \\ &= (x^2 + 4y^2 + 2xy)(x^2 + 4y^2 - 2xy). \\ \therefore x^4 + 4x^2y^2 + 16y^4 &= (x^2 + 2xy + 4y^2)(x^2 - 2xy + 4y^2). \end{aligned}$$

Exercise 88. Factoring

Examples 1 and 2, oral — Examples 3 to 12, written

1. What must be added to the middle term of $x^4 + 3x^2y^2 + 4y^4$ to make the trinomial a perfect square?

2. What must be added to the trinomials $x^4 - 5x^2y^2 + 4y^4$ and $x^4 + 4y^4$ to make them perfect squares?

Factor the following:

3. $p^4 + p^2 + 1$.

4. $x^4 - 7x^2 + 1$.

5. $16a^4 + 4a^2 + 1$.

6. $9a^4 + 26a^2b^2 + 25b^4$.

7. $9m^4 - 15m^2 + 1$.

8. $x^4 + 4$.

9. $x^4 - 7x^2y^2 + y^4$.

10. $9m^4 + 11m^2 + 4$.

11. $4a^4 - 29a^2b^2 + 25b^4$.

12. $625a^4 + 100a^2b^2 + 16b^4$.

Exercise 89. Review and Applications

Examples 1 to 7, oral — Examples 8 to 29, written

1. Factor $ax - x^2$; $a^2 - ax$; $a^2 - x^2$.

2. Factor $a^2 + 2ab + b^2$; $a^2 + 6ab + 9b^2$; $a^2 - 6ab + 9b^2$.

3. Factor $4x^2 + 4x + 1$; $4x^2 - 4x + 1$; $x^2 - 4x + 4$.

4. Factor $a^2 - x^2$; $(a + b)^2 - x^2$; $(a + b)^2 - c^2$.

5. Factor $a^4 - x^4$; $m^4 - n^4$; $p^4 - q^4$; $b^4 - c^4$.

6. Factor $ax + bx + cx$; $ax - bx + cx$; $ax - bx - cx$.

7. Factor $x(a + b) + y(a + b)$; $x(a + b) + x(c + d)$.

Factor the following:

8. $a^4 - 2a^2b^2 + b^4$; $x^4 - 16y^4$; $x^8 - y^8$.

9. $4a^2 - 4a + 1$; $144x^2 - 24xy + y^2$.

10. $1 + 6x + 9x^2$; $4ac + 4c^2 + a^2 - 6x - 1 - 9x^2$.

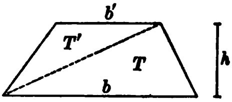
11. $q^2 - 4qm^2 + 4m^4$; $p^4 - 6p^2r + 9r^2 - 4m^4 - q^2 + 4qm^2$.

12. $25x^2 - 9x^2m^2 - 1 - 10xy + y^2 + 6xm$.

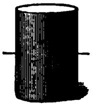
13. $30p^3x + a^4x + 16p^4x - 25p^6x - 9x + 8a^2p^2x$.

14. $4ax^2 - 25ay^2 - ax^4 - 36ax + 81a + 10ax^2y$.

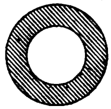
15. In the trapezoid here shown, the area of triangle T is $\frac{1}{2}bh$, and the area of triangle T' is $\frac{1}{2}b'h$. What is the area of the trapezoid? Factor the result, letting $\frac{1}{2}h$ be one factor.



16. In a cylinder the area of each base is πr^2, and the curved surface has an area of $2\pi rh$. This makes the total surface how much? Factor the result, thus simplifying the formula.



17. The area of the outside circle here shown is πa^2, and the area of the inside circle is πb^2. What is the area of the ring formed by the two circles? Factor the result, thus simplifying the formula.



Factor the following:

18. $4x^2 + 12xy + 9y^2 - 9a^2 - 24ab - 16b^2$.

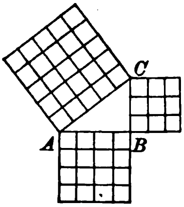
19. $x^2y^2z^2 + 2xyz + 1 - 2m^2n^2z - m^4n^4 - z^2$.

20. $1.21a^2b^2c^2 + 1 - x^2 - y^2 - 2.2abc - 2xy$.

21. $-2.64ab - x^2 + 1.44a^2 - y^4 + 1.21b^2 - 2xy^2$.

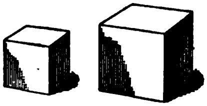
22. $-0.48ab - m^2 + 0.16a^2 - n^2 - 2mn + 0.36b^2$

23. In the figure here shown, we have found that if $AB = b$, $BC = a$, and $AC = h$, then $h^2 = a^2 + b^2$, or $b^2 = h^2 - a^2$. Write the equation $b^2 = h^2 - a^2$ with the second member factored.



24. Using the factored form in Ex. 23, find the value of b^2 when $h = 5$, $a = 3$; when $h = 5$, $a = 4$; when $h = 35$, $a = 28$; when $h = 45$, $a = 36$.

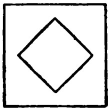
25. The edge of the larger of two cubes is a, and that of the smaller is b. The area of the base of one is how much greater than that of the other? Factor the result and evaluate for $a = 30$, $b = 20$.



26. In Ex. 25 what is the area of the entire surface of the six faces of each cube? What is the difference in area? Factor the result and evaluate for $a = 30$, $b = 20$.

27. The amount of principal and interest on a note is given by the formula $a = p + prt$. Factor the second member and evaluate for $p = 350$, $r = 6\%$, $t = 3$.

28. From a square of side a is cut a square of side b. What is the area of the remaining part? Factor the result and evaluate for $a = 60$, $b = 20$.



29. The area of the surface of a sphere of radius r is $4\pi r^2$. What is the difference of areas of two spheres of radii a and b, respectively? Factor the result and evaluate for $a = 5$, $b = 4$, and $\pi = 3\frac{1}{7}$.

112. Factoring the Quadratic Trinomial. A trinomial of the form $x^2 + bx + c$ is called a *quadratic trinomial.*

The more general form, $ax^2 + bx + c$, is considered in § 114.

To find the method of factoring this trinomial we consider the product of two binomials.

$$\begin{array}{lll} x & +\ 3 & \\ x & +\ 7 & \\ \hline x^2 & +\ 3x & \\ & \ 7x & +\ 21 \\ \hline x^2 & +\ 10x & +\ 21 \end{array} \qquad\qquad \begin{array}{lll} x & +\ a & \\ x & +\ b & \\ \hline x^2 & +\ ax & \\ & \ bx & +\ ab \\ \hline x^2 & +(a+b)x & +\ ab \end{array}$$

We see that the factors of $x^2 + 10x + 21$ are $x + 3$ and $x + 7$, and that the factors of $x^2 + (a + b)x + ab$ are $x + a$ and $x + b$.

Hence, if a trinomial of the form $x^2 + px + q$ is factorable, the first term of each factor will be x; and the second terms of the factors will be two numbers whose product is q and whose algebraic sum is p, the coefficient of x.

1. Factor $x^2 + 12x + 35$.

The two numbers whose product is 35 and whose sum is 12 are evidently 5 and 7.

Therefore $x^2 + 12x + 35 = (x + 5)(x + 7)$.

This may be checked by multiplying, or by substituting any convenient value for x.

2. Factor $x^2 + 5x - 36$.

Since the product of the second terms is -36, one must be positive and the other negative.

The algebraic sum of the two numbers is $+5$, hence the positive number must have the greater numerical value.

The two numbers whose product is -36 and whose algebraic sum is 5 are evidently 9 and -4.

Therefore $x^2 + 5x - 36 = (x + 9)(x - 4)$.

3. Factor $x^2 + 0.3x + 0.02$.

The two numbers whose product is 0.02 and whose sum is 0.3 are evidently 0.1 and 0.2.

Therefore $x^2 + 0.3x + 0.02 = (x + 0.1)(x + 0.2)$.

4. Factor $x^2 - 5x - 14$.

The absolute term, which is the product of the two numbers sought, being negative, the two numbers have unlike signs.

The algebraic sum being negative, the negative number has the greater numerical value.

The two numbers whose product is -14 and whose sum is -5 are -7 and $+2$.

Therefore $x^2 - 5x - 14 = (x - 7)(x + 2)$.

5. Factor $x^2 - 12x + 27$.

The product being positive, the two numbers have like signs.

The sum being negative, both numbers are negative.

The two numbers whose product is $+27$ and whose sum is -12 are -9 and -3.

Therefore $x^2 - 12x + 27 = (x - 9)(x - 3)$.

6. Factor $x^2 - 3xy - 28y^2$.

The second terms have unlike signs. Why?

The negative term has the greater numerical value. Why?

The second terms are $-7y$ and $+4y$. Why?

The factors are $(x - 7y)(x + 4y)$.

7. Factor $x^2 + 0.2x - 0.03$.

What are signs of the second terms? Which of these terms is numerically the greater? What are the second terms? What are the factors?

113. Directions for factoring $x^2 + bx + c$. We see therefore that in factoring an expression of the form $x^2 + bx + c$, we proceed as follows:

Find two monomials whose algebraic product is the absolute term with its proper sign, and whose algebraic sum is the coefficient of x with its proper sign.

Write for the factors two binomials, the first term of each being x, and the second terms being, respectively, the monomials thus found.

We should notice that when c is negative the two monomials sought have unlike signs.

When c is positive the monomials have each the same sign as the middle term, bx.

Exercise 90. Factoring the Quadratic Trinomial

Examples 1 to 10, oral — Examples 11 to 38, written

1. Find two numbers whose sum is 8 and whose product is 15; whose sum is 2 and whose product is -15.

2. Find two numbers whose sum is -2 and whose product is -15; whose sum is 2 and whose product is -35.

Find two numbers whose sum s and product p are:

3. $s = 8, p = 12$.
4. $s = 10, p = 24$.
5. $s = -5, p = -14$.
6. $s = -5, p = -6$.
7. $s = -1, p = -6$.
8. $s = -13, p = 40$.
9. $s = -16, p = 63$.
10. $s = -13, p = 22$.

Factor the following:

11. $a^2 + 8a + 15$.
12. $a^2 + 10a + 24$.
13. $x^2 - 5x - 14$.
14. $x^2 - 5x - 6$.
15. $p^2 - p - 6$.
16. $p^2 - 13p + 40$.
17. $n^2 - 16n + 63$.
18. $n^2 - 13n + 22$.
19. $m^2 - 2m - 15$.
20. $b^2 + 2b - 24$.
21. $b^2 + 5b - 14$.
22. $t^2 + 5t - 6$.
23. $x^2 + 5xy - 36y^2$.
24. $x^2 + xy - 12y^2$.
25. $x^2 - 4xy - 21y^2$.
26. $x^2 + xy - 72y^2$.
27. $x^2 + xy - 56y^2$.
28. $x^2 + 23x + 132$.
29. $x^2 + 3x - 130$.
30. $x^2 - 2x - 143$.
31. $x^2 + 5x - 150$.
32. $x^2 - 4x - 165$.
33. $x^2 + 30xy + 200y^2$.
34. $x^2 - 10xy - 200y^2$.

Factor both terms and then reduce to lowest terms:

35. $\dfrac{x^2 + 8x + 15}{x^2 + x - 20}$.
36. $\dfrac{a^2 + 10a + 24}{a^2 - a - 42}$.
37. $\dfrac{p^2 + 20p + 99}{p^2 + 21p + 108}$.
38. $\dfrac{x^2 + 22x + 121}{x^2 + 23x + 132}$.

114. Factoring the General Quadratic Trinomial. There is a more general form of the quadratic trinomial than that studied in § 112. For example, we may have the trinomial $6x^2 + 17x + 12$, or, more generally, $ax^2 + bx + c$.

There are several acceptable methods of factoring such expressions. The one requiring least explanation is given below; but, if the teacher prefers, one of those given in the Appendix may be used instead.

If we consider the product of $3x + 4$ and $2x + 3$ we see that the first term of the product $(6x^2)$ is the product of the first terms of the factors, $3x$ and $2x$; that the last term of the product (12) is the product of the last terms of the factors, 4 and 3; and that the middle term of the product $(17x)$ is the sum of the products of the first term of either factor and the last term of the other factor, that is, the sum of their "cross products."

$$\begin{array}{rrr} 3x & + \;4 & \\ 2x & + \;3 & \\ \hline 6x^2 & + \;8x & \\ & 9x & + 12 \\ \hline 6x^2 & + 17x & + 12 \end{array}$$

Therefore, *in factoring an expression of the form* $ax^2 + bx + c$, *find two binomials such that the product of the first terms is the first term of the trinomial, and the product of the second terms is the last term of the trinomial, and such that the algebraic sum of their cross products is the middle term. These binomials are the required factors.*

As usual, the monomial factors are first to be removed.

1. Factor $6x^2 + 17x + 12$.

The factors of $6x^2$ are $3x$ and $2x$, or $6x$ and x.

The factors of 12 are 4 and 3, 6 and 2, or 12 and 1.

We must so select that the product of the first factor of $6x^2$ and the last of 12, and the product of the last factor of $6x^2$ and the first of 12, shall have as their algebraic sum $17x$.

We see that we cannot take $3x + 3$, because this contains the factor 3, which is not a factor of $6x^2 + 17x + 2$.

Similarly, we may reject $2x + 6$, $6x + 12$, $3x + 12$, $3x + 6$, and all other binomials that contain a monomial factor.

After rejecting these impossible factors we find, by a little trial, that $3x + 4$ and $2x + 3$ are the factors. We check by substituting some value for x, or by multiplication.

2. Factor $15x^2 - 11x - 14$.

The factors of $15x^2$ are $5x$ and $3x$, or $15x$ and x.

The factors of 14 are 7 and 2, or 14 and 1; and since we have -14, one of these must be negative.

By placing the numbers as shown at the right we can easily see that $5 \cdot 2 - 3 \cdot 7 = -11$, and hence that the factors are $5x - 7$ and $3x + 2$.

$$\begin{array}{cc} 5 & 7 \\ 3 & 2 \end{array}$$

Check. Letting $x = 1$ we have $5x - 7 = -2$, $3x + 2 = 5$, and $-2 \cdot 5 = -10$. Also, $15x^2 - 11x - 14 = -10$.

We may also check by multiplying.

3. Factor $77x^2 + 41xy - 10y^2$.

We evidently have $(7x \pm 5y)(11x \mp 2y)$, where the upper signs go together and the lower also go together; or we may have $(7x \pm 10y)(11x \mp y)$, and so on.

Arranging the numbers as in Ex. 2, we easily see that $(7x + 5y)(11x - 2y)$ give the required product.

$$\begin{array}{cc} 7 & 5 \\ 11 & -2 \end{array}$$

We check the work as in Ex. 2.

4. Factor $6x^2 - 77x + 221$.

The factors of $6x^2$ are $3x$ and $2x$, or $6x$ and x.

The factors of 221 are 13 and 17, or 221 and 1, and both must be negative to have $-77x$ and $+221$.

Evidently -1 and -221 cannot be used, for they would make the second term too large.

Of $(3x - 13)(2x - 17)$ or $(3x - 17)(2x - 13)$, it is easily seen the first gives the required product.

5. Factor $32x^2 + 867x + 81$.

The factors of $32x^2$ are $32x$ and x, $16x$ and $2x$, or $8x$ and $4x$.

The factors of 81 are 81 and 1, 27 and 3, or 9 and 9.

The middle term tells us that we must choose large factors, but evidently not 32 with 81, or 16 with 27, because, as here shown, the sum of the cross products would not be large enough.

$$\begin{array}{cc} 32 & 81 \\ 1 & 1 \end{array}$$

By a little thought and by one or two trials, we see that the factors are $32x + 3$ and $x + 27$.

6. Factor $12x^3 - 34x^2 - 28x$.

We first take out the factor $2x$.

We then have to factor $2x(6x^2 - 17x - 14)$.

The factors of $6x^2 - 17x - 14$ are $3x + 2$ and $2x - 7$.

Therefore $12x^3 - 34x^2 - 28x = 2x(3x + 2)(2x - 7)$.

Exercise 91. Factoring the General Quadratic Trinomial

Examples 1 to 5, oral — Examples 6 to 41, written

1. In factoring $4x^2 + 8x + 3$, how will you proceed?

2. In factoring $64x^2 + 288x + 243$, how will you proceed?

3. In factoring $12x^2 - 19x - 21$, what do you know in advance as to the signs of the absolute terms of the factors?

4. In factoring $56x^2 - 22x + 2$, what do you know in advance as to the signs of the absolute terms of the factors?

5. After factoring $42x^2 - 85x + 42$, how will you check your work? Give two methods.

Factor the following:

6. $4x^2 + 8x + 3$.
7. $2x^2 + 5x + 3$.
8. $3x^2 - x - 2$.
9. $5x^2 - 8x + 3$.
10. $6x^2 + 7x + 2$.
11. $6x^2 - x - 2$.
12. $15x^2 + 14x - 8$.
13. $8x^2 - 10x + 3$.
14. $18x^2 + 9x - 2$.
15. $12x^2 - 5x - 2$.
16. $12x^2 - 7x + 1$.
17. $12x^2 - x - 1$.
18. $3x^2 - 2x - 5$.
19. $3x^2 + 4x - 4$.
20. $6x^2 + 5x - 4$.
21. $4x^2 + 13x + 3$.
22. $4x^2 + 11x - 3$.
23. $4x^2 - 4x - 3$.
24. $4x^2 + 16x + 7$.
25. $6x^2 + 17x + 12$.
26. $6x^2 + 17ax + 12a^2$.
27. $6a^2x^2 + 17axy + 12y^2$.
28. $6x^2y^2 + xy - 1$.
29. $12x^2 - 13xy - 14y^2$.
30. $10a^2 - 23ab - 5b^2$.
31. $8p^2 + 53pq - 21q^2$.
32. $8m^2 - 37mn - 15n^2$.
33. $2x^2 + 5xy + 2y^2$.
34. $8x^2 - 5bx - 3b^2$.
35. $8a^2 + 14ab - 15b^2$.
36. $60x^2 + 99x + 21$.
37. $45x^2 + 81x + 28$.
38. $125a^2b^2 + 135abc + 28c^2$.
39. $102a^2x^2 + 101axy - 21y^2$.
40. $130p^2q^2 + 21r^2 - 109pqr$.
41. $1 + 209x^2y^2z^2 - 30xyz$.

115. Factoring the Sum or Difference of Two Cubes. We have found (§ 97) that the sum of the cubes of two quantities is divisible by the sum of the quantities. Therefore, since

$$\frac{a^3+b^3}{a+b}=a^2-ab+b^2,$$

$$\mathbf{a^3+b^3=(a+b)(a^2-ab+b^2).}$$

Similarly, $$\mathbf{a^3-b^3=(a-b)(a^2+ab+b^2).}$$

Therefore, *the factors of the sum of the cubes of two quantities are* (1) *the sum of the quantities;* (2) *the square of the first, minus the product of the first and second, plus the square of the second.*

The factors of the difference of the cubes of two quantities are (1) *the difference of the quantities;* (2) *the square of the first, plus the product of the first and second, plus the square of the second.*

1. Factor $27x^3+y^3$.

$$\begin{aligned}27x^3+y^3&=(3x)^3+y^3\\&=(3x+y)(9x^2-3xy+y^2).\end{aligned}$$

2. Factor $8x^3+27y^3$.

$$\begin{aligned}8x^3+27y^3&=(2x)^3+(3y)^3\\&=(2x+3y)(4x^2-6xy+9y^2).\end{aligned}$$

3. Factor $27a^3-8b^3$.

$$\begin{aligned}27a^3-8b^3&=(3a)^3-(2b)^3\\&=(3a-2b)(9a^2+6ab+4b^2).\end{aligned}$$

4. Factor $(a-b)^3+c^3$.

$$\begin{aligned}(a-b)^3+c^3&=(a-b+c)[(a-b)^2-(a-b)c+c^2]\\&=(a-b+c)(a^2-2ab+b^2-ac+bc+c^2).\end{aligned}$$

5. Factor p^9+q^{12}.

Since $p^9=p^3p^3p^3$, therefore $p^9=(p^3)^3$. Similarly, $q^{12}=(q^4)^3$

Therefore, $$\begin{aligned}p^9+q^{12}&=(p^3)^3+(q^4)^3\\&=(p^3+q^4)[(p^3)^2-p^3q^4+(q^4)^2]\\&=(p^3+q^4)(p^6-p^3q^4+q^8),\end{aligned}$$

because $(p^3)^2=p^3p^3=p^6$, and $(q^4)^2=q^4q^4=q^8$ (§ 63).

Exercise 92. The Sum or Difference of Cubes

Examples 1 to 9, oral — Examples 10 to 41, written

1. Factor a^3+b^3; a^3+1; x^3+1; a^3+2^3; m^3+3^3; p^3+q^3.

2. Factor a^3-b^3; x^3-1; m^3-2^3; p^3-3^3; a^3-8; x^3-27.

3. Factor m^3+1; m^3-1; x^3y^3+1; x^3y^3-1; $1+x^3y^3$.

4. Factor $1+a^3$; $1+x^3$; $1+m^3$; $1+p^3$; $1+q^3$; $1+y^3$.

5. Factor $1-a^3$; $1-x^3$; $1-m^3$; $1-p^3$; $1-q^3$; $1-y^3$; m^3n^3-1; $1-m^3n^3$; $a^3b^3c^3-1$; $1-x^3y^3$.

Factor the following:

6. $8x^3+1$.

7. $1+8x^3$.

8. $27x^3+1$.

9. $1+27x^3$.

10. $27x^3+8$.

11. $8x^3-27$.

12. $27-8x^3$.

13. $27x^3-125$.

14. $125-27x^3$.

15. $27x^3-8a^3$.

16. m^3+8n^3.

17. $27m^3-n^3$.

18. $64x^3-1$.

19. $1-125x^3$.

20. $216+x^3$.

21. Represent x^6+y^9 as the sum of the cubes of the two quantities, x^2 and y^3, and factor the result.

22. Represent $x^{12}-1$ as the difference of two cubes and factor the result.

Factor the following:

23. x^9-y^6.

24. $x^{12}+y^6$.

25. $x^{12}-y^9$.

26. $8x^{12}+1$.

27. $1-27x^6$.

28. $8-27x^9$.

29. $x^{12}+8y^{12}$.

30. $x^{15}-8$.

31. $x^{15}+y^9$.

32. $x^{15}-y^{12}$.

33. $8x^{15}+y^6$.

34. $27x^{15}-1$.

35. Represent a^6b^6+1 as the sum of two cubes and factor the result.

Factor the following:

36. a^3b^3+512.

37. $a^3b^3-512(x+y)^3$.

38. $(a+b)^3+512(x+y)^3$.

39. $729+(a+b)^3$.

40. $(a+b)^3+(c+d)^3$.

41. $(x+y)^3-(a+b)^3$.

116. Factoring Perfect Cubes. We have found (§ 95) that

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

These statements may be combined, thus:

$$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3.$$

When there is the double sign, ±, in both members, the upper signs go together and the lower signs go together.

Therefore *the cube root of a polynomial in the form* $a^3 \pm 3a^2b + 3ab^2 \pm b^3$ *is of the form* $a \pm b$.

Factor $8x^3 + 12x^2 + 6x + 1$; that is, find its cube root.

Since this polynomial equals $(2x)^3 + 3(2x)^2 + 3(2x) + 1$, it equals $(2x+1)^3$. The factors are therefore $2x+1$, $2x+1$, and $2x+1$. The answer should be written in the form $(2x+1)^3$.

Exercise 93. Factoring Perfect Cubes

Examples 1 to 6, oral — Examples 7 to 21, written

1. Factor $x^3 + 3x^2y + 3xy^2 + y^3$; $x^3 - 3x^2y + 3xy^2 - y^3$.
2. Factor $a^3 + 3a^2 + 3a + 1$; $a^3 - 3a^2 + 3a - 1$.
3. Factor $1 - 3m + 3m^2 - m^3$; $1 + 3xy + 3x^2y^2 + x^3y^3$.

Factor the following:

4. $2^3 + 3 \cdot 2^2 + 3 \cdot 2 + 1$.
5. $3^3 + 3 \cdot 3^2 + 3 \cdot 3 + 1$.
6. $7^3 + 3 \cdot 7^2 + 3 \cdot 7 + 1$.
7. $p^3 + 6p^2 + 12p + 8$.
8. $x^3 + y^3 + 3xy(x+y)$.
9. $a^3 + 6a^2 + 12a + 8$.
10. $a^3 - 6a^2 + 12a - 8$.
11. $8a^3 + 12a^2 + 6a + 1$.
12. $8a^3 - 12a^2 + 6a - 1$.
13. $8a^3b^3 + 12a^2b^2 + 6ab + 1$.
14. $1 - 6ab + 12a^2b^2 - 8a^3b^3$.
15. $8a^3b^3 + 12a^2b^2c + 6abc^2 + c^3$.
16. $8a^3b^3 - 12a^2b^2c + 6abc^2 - c^3$.
17. $27a^3 + 27a^2 + 9a + 1$.
18. $1 - 9a + 27a^2 - 27a^3$.
19. $64x^3 + 48x^2 + 12x + 1$.
20. $64x^3 + 12x - 48x^2 - 1$.
21. $x^3 - 12x^2y + 48xy^2 - 64y^3$

117. The Remainder Theorem. If we divide $x^2 - 7x + 10$ by $x - 2$ the quotient is $x - 5$ and there is no remainder. Furthermore, if we put 2 for x in $x - 2$ we have $2 - 2 = 0$; and if we put 2 for x in the polynomial we have $2^2 - 7 \cdot 2 + 10 = 0$. That is, the remainder and the polynomial both become zero when we put 2 in place of x.

If we divide $x^2 - 7x + 4$ by $x - 2$, there is a remainder -6, and if we put 2 for x in the function we have $2^2 - 7 \cdot 2 + 4 = -6$.

In general, if we divide $x^2 + bx + c$ by $x - a$ we see that the remainder is $a^2 + ba + c$, which is the same as the dividend with a substituted for x. That is,

$$\begin{array}{r|l}
x^2 + bx + c & x - a \\ \cline{2-2}
x^2 - ax & x + a + b \\ \cline{1-1}
(a + b)x + c & \\
(a + b)x - a^2 - ba & \\ \cline{1-1}
a^2 + ba + c &
\end{array}$$

The remainder arising from dividing any integral function of x by $x - a$ is the same expression with a put in place of x.

This is called the Remainder Theorem, the word "theorem" meaning a statement to be proved.

Expressed in the symbols of functions, the remainder arising from dividing $f(x)$ by $x - a$ is $f(a)$.

This important theorem enables us to factor many expressions that do not come under the cases already considered, or to factor the latter more easily.

1. Is $x - 1$ a factor of $x^3 - 7x + 6$?

Substitute 1 for x and we have $1 - 7 + 6 = 0$. Therefore there is no remainder when we divide this $f(x)$ by $x - 1$. Therefore $x - 1$ is a factor.

2. Is $x - 3$ a factor of $x^2 - 9x + 6$?

Substitute 3 for x and we have $9 - 27 + 6 = -12$.
Therefore if we divide this $f(x)$ by $x - 3$ there is a remainder -12. Therefore $x - 3$ is not a factor.

3. Is $x + 2$ a factor of $x^3 + 4x^2 + 3x - 2$?

Since $x + 2 = x - (-2)$, which is now in the form $x - a$, substitute -2 for x and we have $(-2)^3 + 4(-2)^2 + 3(-2) - 2 = -8 + 16 - 6 - 2 = 0$. Therefore $x + 2$ is a factor.

4. Find the factors of $x^3 + 4x^2 - 11x - 30$.

Since the factors of -30 are $\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15$, we try binomial factors of which these are the second terms.

Trying $x - 1$, $f(1) = 1 + 4 - 11 - 30$, not zero.
Trying $x + 1$, $f(-1) = -1 + 4 + 11 - 30$, not zero.
Trying $x - 2$, $f(2) = 8 + 16 - 22 - 30$, not zero.
Trying $x + 2$, $f(-2) = -8 + 16 + 22 - 30 = 0$.

Therefore $x + 2$ is one factor. Dividing by $x + 2$ we have $x^2 + 2x - 15$, of which we see the factors are $x + 5$ and $x - 3$.

Therefore the factors are $x + 2$, $x - 3$, $x + 5$.

Exercise 94. The Remainder Theorem

Examples 1 to 6, oral — Examples 7 to 21, written

1. Is $x - 1$ a factor of $x^3 - x^2 + x - 1$? of $x^3 - 2x^2 + x$?

2. Is $x - 1$ a factor of $x^2 - 1$? of $x^4 - 1$? of $x^{10} - 1$? of $x^3 - 4x^2 + 4x - 1$? of $x^3 + 7x^2 - 9x + 1$?

3. Is $x - 1$ a factor of $x^{57} - 1$? of $x^{100} - 1$? of $x^{25} + x^{13} - 2$?

4. What remainder arises from dividing $x^{25} + 1$ by $x - 1$?

5. What remainder arises from dividing $x^7 + x^5 + 1$ by $x - 1$? $x^{19} - 2x^{17} + 1$ by $x - 1$?

6. Is $x + 1$ a factor of $x^3 + 1$? of $x^5 + 1$? of $x^7 + 1$? of $x^9 + 1$? of $x^{11} + 1$? of $x^3 + 8$? of $x^2 + 1$?

7. Is $x - 2$ a factor of $x^2 - 4$? of $x^4 - 16$? of $x^6 - 64$? of $x^3 - 9x^2 + 17x - 6$? of $x^3 - 8$? of $x^3 + 8$?

Factor the following:

8. $x^3 - 7x - 6$.
9. $a^3 - 8a + 3$.
10. $x^3 + 7x^2 - 8$.
11. $m^3 - 19m + 12$.
12. $x^3 + 2x^2 - 9x - 18$.
13. $x^3 + 2x^2 - 2x - 1$.
14. $x^3 + x^2 - 4x - 4$.
15. $x^3 - 5x^2 - 2x + 24$.
16. $a^3 - 48ab^2 - 7b^3$.
17. $m^3 - 9mn^2 + 8n^3$.
18. $3p^3 + 4pq^2 - 7q^3$.
19. $x^3 - x^2 - 8x + 12$.
20. $a^3 - 9a^2 + 26a - 24$.
21. $x^3 - 10x^2 + 23x - 14$.

118. Factors of $x^n \pm y^n$. The Remainder Theorem enables us to determine the cases in which $x^n \pm y^n$ is divisible by $x \pm y$.

1. Is $x^n + y^n$ divisible by $x - y$?

Substituting y for x in the binomial $x^n + y^n$ we have $y^n + y^n = 2y^n$. Hence $x^n + y^n$ divided by $x - y$ has a remainder $2y^n$. Hence $x^n + y^n$ is not divisible by $x - y$, whatever may be the value of n.

2. Is $x^n - y^n$ divisible by $x - y$?

Substituting y for x in the binomial $x^n - y^n$ we have $y^n - y^n = 0$. Therefore *$x - y$ is always a factor of $x^n - y^n$.*

In general, in factoring expressions like $x^6 - y^6$ it is better first to take two factors of the same degree, thus:

$$x^6 - y^6 = (x^3 + y^3)(x^3 - y^3) = (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2).$$

This is better than to start with

$$x^6 - y^6 = (x - y)(x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5),$$

for the factors of the second of these factors are not readily seen.

3. Is $x^n + y^n$ divisible by $x + y$?

Since $x + y = x - (-y)$, and this is in the form $x - a$, we substitute $-y$ for x in $x^n + y^n$, and $x^n + y^n$ becomes $(-y)^n + y^n$.

If n is odd $(-y)^n$ is negative (§ 64) and equals $-y^n$. We then have $-y^n + y^n = 0$. That is, *if n is odd $x + y$ is a factor of $x^n + y^n$.*

If n is even $(-y)^n$ is positive and equals y^n. We then have $y^n + y^n = 2y^n$. That is, *if n is even $x + y$ is not a factor of $x^n + y^n$.*

For example, $x + y$ is a factor of $x^3 + y^3$, but not of $x^2 + y^2$.

Similarly, $x + 2$ is a factor of $x^3 + 8$, but not of $x^4 + 16$.

4. Is $x^n - y^n$ divisible by $x + y$?

Substituting $-y$ for x we have $(-y)^n - y^n$. If n is even this becomes $y^n - y^n = 0$, because $-y$ raised to any even power is positive (§ 64). But if n is odd it becomes $-y^n - y^n = -2y^n$, because $-y$ raised to any odd power is negative (§ 64). That is, *if n is even $x + y$ is a factor of $x^n - y^n$, but not if n is odd.*

Summarizing these cases we have the following:

$x^n + y^n$ never has a factor $x - y$;

$x^n - y^n$ always has a factor $x - y$;

$x^n + y^n$ has a factor $x + y$ when n is odd;

$x^n - y^n$ has a factor $x + y$ when n is even.

119. Summary of Factoring. The student will find it of advantage to proceed as follows in factoring an expression:

First remove any monomial factor. Then see if the expression is one of the following forms already studied:

$ax + ay + bx + by.$ (§ 104)
$a^2 \pm 2ab + b^2.$ (§ 105)
$a^2 - b^2.$ (§§ 107–111)
$x^2 + bx + c.$ (§ 112)
$ax^2 + bx + c.$ (§ 114)
$a^3 \pm b^3.$ (§ 115)
$a^3 \pm 3a^2b + 3ab^2 \pm b^3.$ (§ 116)
$(a - b) \times$ a polynomial. (§§ 117, 118)

If so, factor as directed under these cases.

Factor each polynomial of the result until every factor is prime.

Students are urged to check their results in factoring, either by the substitution of some value for the letters or by multiplication, and teachers may well afford to insist upon it.

120. Changing the Signs of Factors. Since $abc = a \cdot -b \cdot -c = -a \cdot b \cdot -c = -a \cdot -b \cdot c$, we see that

The signs of any even number of factors may be changed without changing their product.

Therefore $a^2 - 2ab + b^2 = (a - b)(a - b)$ or $(b - a)(b - a)$.

Exercise 95. Review of Factoring

Examples 1 to 5, oral—Examples 6 to 142, written

1. Factor $ax + bx$; $ax - ay$; $am + am^2$; $ax + bx - cx$.
2. Factor $a^2 + 2a + 1$; $a^2 - 2a + 1$; $1 - 2a + a^2$.
3. Factor $m^2 - 1$; $1 - x^2$; $a^2b^2 - 1$; $a^2b^2 - c^2$.
4. Factor $a^3 + b^3$; $a^3 - b^3$; $a^3 + 1$; $1 + a^3$; $a^3 - 1$; $1 - a^3$.
5. Factor $a^3 + 3a^2 + 3a + 1$; $a^3 - 3a^2 + 3a - 1$.

Factor the following:

6. $x^4 - x^2$.
7. $x^4 - y^2$.
8. $x^4 - 1$.
9. $x^4 - y^4$.
10. $x^9 + y^9$.
11. $x^9 + 8\,y^9$.
12. $x^6 - y^6$.
13. $1 - y^6$.
14. $49\,a^2 - 16\,b^2$.
15. $49\,a^2 - (c + d)^2$.
16. $(a - b)^2 - 36\,c^2$.
17. $a^2b^2 - (c - d)^2$.
18. $1 - (c - d)^2$.
19. $(a + b)^3 + 1$.
20. $1 - (a - b)^3$.
21. $(x - y)^3 - 8$.
22. $x^2 + 14\,x + 49$.
23. $x^2 + 64 - 16\,x$.
24. $1 + 4\,a^2b^2 - 4\,ab$.
25. $a^2 - 11\,a + 18$.
26. $p^2 + 9\,p - 36$.
27. $4\,a^2b^2 + 12\,abcd + 9\,c^2d^2$.
28. $9\,x^2y^2 + 4 - 12\,xy$.
29. $(a + b)^2 + 1 - 2\,(a + b)$.
30. $1 + (2x + y)^2 + 2\,(2x + y)$.
31. $m^2n^2 - 24\,mn + 95$.
32. $m^2n^4 + 24\,mn^2 + 95$.
33. $m^2n^6 + 95\,n^2 - 24\,mn^4$.
34. $a^2 + 2\,ab - c^2 + b^2$.
35. $a^2 - b^2 - c^2 + 2\,bc$.
36. $(p - q)^2 - (x - y)^2$.
37. $1 - x^2 - y^2 + 2\,xy$.
38. $1 - x^2 - 4\,y^2 - 4\,xy$.
39. $m^2 + 2\,mn + 3\,pm + 6\,np$
40. $9\,a^2 + 30\,ab + 25\,b^2$.
41. $9\,a^4 + 21\,a^2b^2 + 25\,b^4$.
42. $5\,a^2 + 13\,a - 6$.
43. $15\,a^2 + 19\,a + 6$.
44. $9\,a^2 + 9\,a + 2$.
45. $3\,p^2q^2 + 7\,pq - 6$.
46. $3\,p^2 + 7\,pq - 6\,q^2$.
47. $6\,a^2 + 7\,ab + 2\,b^2$.
48. $6\,a^2 - ab - b^2$.
49. $12\,a^2 - 7\,ab + b^2$.
50. $12\,a^2 + 5\,ab - 2\,b^2$.
51. $8\,a^2 + 2\,ab - b^2$.
52. $49\,x^4 - 81\,y^2$.
53. $a^2 + 2\,ab - 99\,b^2$.
54. $m^6n^6 + 8\,t^3$.
55. $49\,m^2n^2 - 121\,p^2$.
56. $121\,x^2y^2 - 36\,z^2$.
57. $x^2 - 20\,xy + 51\,y^2$.
58. $64\,a^2b^2 - 49\,c^4d^4$.
59. $729\,x^3 - y^3$.
60. $729\,x^3 - 343$.
61. $x^3 - 5\,x^2 - 84\,x$.
62. $32\,x^2 + 4\,xy - 15\,y^2$.
63. $81\,x^4 - 625\,y^4$.

Factor the following:

64. $a^2 - 16ab^2 + 60b^4$.

65. $27m^3 - 1331$.

66. $27m^3 - 1331x^3y^3$.

67. $a^4 + 5a^3 - 36a^2$.

68. $625x^4y^4 - 81$.

69. $81x^4y^4 - 625$.

70. $a^2 + c - c^2 - a$.

71. $a^3 + a - b - b^3$.

72. $(a + b)^3 - 1$.

73. $1 - (a - b)^3$.

74. $x^{2n} - y^{2n}$.

75. $9x^2 - 24x + 16$.

76. $a^3 + b^2 - a^2 - b^3$.

77. $a^2 + a^4 - b^2 - b^4$.

78. $24m^2 - 35n^2 + 2mn$.

79. $m^2n^2 - 7mn - 44$.

80. $p^2 + 30q^2 + 13pq$.

81. $4a^4 + 8a^2b^2 + 121b^4$.

82. $2ax + 4bx - 10pb - 5pa$.

83. $p^4q^4 - x^4 + p^2q^2 - x^2$.

84. $p^4q^4 - p^2q^2 - x^4 + x^2$.

85. $81a^2 - 180ab + 100b^2$.

86. $x^2y^4 - 12xy^2z + 35z^2$.

87. $64m^2 - 48mn + 9n^2$.

88. $24x^4 - 86x^2y^2 + 42y^4$.

89. $p^2 + 8pq^2 - 33q^4$.

90. $6m^4 + 25m^2n^2 - 91n^4$.

91. $ac - bc - ad + bd$.

92. $xy - xz - y^2 + yz$.

93. $ab - 2a - 2b + 4$.

94. $49x^2 + 56xy + 16y^2 - 16z^2$.

95. $49a^5b - 154a^3b^3 + 121ab^5$.

96. $20ax + 6by - 15ay - 8bx$.

97. $16a^4c^2 + 88a^3c^3 + 121a^2c^4$.

98. $729x^3 + (y + 2z)^3$.

99. $a^4b^2y + a^3b^3x + ab^4z$.

100. $81x^4 - 18x^2y^2 + 49y^4$.

101. $1728x^3 - 1$.

102. $125x^3 - 1728$.

103. $9x^2 - 24xy^4 + 16y^8$.

104. $a^2 - 25b^2 + 9x^2 + 6ax$.

105. $a^3 + a - b^3 - b$.

106. $x^2y^2 + xy - z^2 - z$.

107. $99a^2b^2 - 17abc - 12c^2$.

108. $p^2q^2 - 12pqr + 35r^2$.

109. $x^2y^2z^2 - a^2 + xyz - a$.

110. $x^2y^2z^2 - xyz - a^2 + a$.

111. $x^2y^2z^2 - 15xyz + 56$.

112. $6x^2y^2z^2 - 37xyz + 56$.

113. $6x^2y^2z^2 - xyz - 35$.

114. $6x^2y^2z^2 + 35 - 29xyz$.

115. $35 + 35p^2q^2r^2 - 74pqr$.

116. $86m^2n^2 + 21m^4n^4 + 33$.

117. $14x^2y^2z^2 - 33 - 71xyz$.

118. If $f(a) = a^2 + 12a + 35$, find the factors of $f(a)$; of $f(a) - 24a$; of $f(a) - 35$; of $f(a) + 4a + 29$.

Arrange as the difference of two squares, and factor:

119. $a^2 - x^2 - y^2 + 4ab + 2xy + 4b^2$.

120. $a^2 + b^4 - x^4 + 2ab^2 - y^4 + 2x^2y^2$.

121. $4a^2 - 9x^2 + 9b^2 - 16y^2 - 12ab + 24xy$.

122. $49a^2 + 1 - c^4 - 14a - 16b^2 - 8bc^2$.

123. $36(a^2 - c^2) + 12(ab^2 - cd^2) + b^4 - d^4$.

124. $9a^2 - 49d^2 + 4b^2 - 16c^2 + 4(3ab + 14cd)$.

125. $4a^4 + 9b^4 - x^4 - 49 + 12a^2b^2 + 14x^2$.

Arrange as the sum or the difference of two cubes, and factor:

126. $a^3 + 3a^2b + 3ab^2 + b^3 - c^3$.

127. $a^3 - b^3 + c^3 + 3b^2c - 3bc^2$.

128. $8a^3 + 1 - b^3 + 12a^2 + 6a$.

129. $a^3 + b^3 - c^3 - d^3 + 3a^2b - 3c^2d + 3ab^2 - 3cd^2$.

130. $a^3 - c^3 + 8b^3 - 8d^3 + 6a^2b - 6c^2d + 12ab^2 - 12cd^2$.

131. $64a^3 - 48a^2 + 12a - 1 - 125b^3 - 75b^2c - 15bc^2 - c^3$.

Factor the following:

132. $144x^2 - 144ac - 81a^2 - 64c^2$.

133. $x^2 - 4n^2 + 4mn - 6xy - m^2 + 9y^2$.

134. $9x^2 - 9a^2 + 64y^2 - 64b^2 + 48xy + 48ab$.

135. $16x^4 - 81b^2 - 25a^4 + 56x^2y - 90a^2b + 49y^2$.

136. $6a^2 - 9ay - 15by - 20bx - 12ax + 10ab$.

137. $25a^2 - 36x^2 + 121b^2 - 49y^2 - 110ab - 84xy$.

138. $(a + b)^2 - 2(a + b)(c + d) + (c + d)^2$.

139. $(a - b)^2 - 2(a - b)(4c - d) + (4c - d)^2$.

140. $x^3 + 6x^2 + 11x + 6$; also $x^3 - 6x^2 + 11x - 6$.

Factor both terms and then reduce to lowest terms:

141. $\dfrac{x^2 - 4xy + 4y^2}{4y^2 - x^2}$.

142. $\dfrac{(4a - 12)^3}{3a^4 - 27a^2}$.

121. Common Factor. A factor of two or more expressions is called a *common factor* of the expressions.

For example, 2 is a common factor of 12 and 16, and $a - b$ is a common factor of $a^2 - b^2$ and $a^4 - b^4$. Similarly, $a - b$ is a common factor of $b - a$ and $a^2 - b^2$, for $(b - a) \div (a - b) = -1$, and $(a^2 - b^2) \div (a - b) = a + b$.

122. Highest Common Factor. The factor of highest degree that is common to two or more expressions is called their *highest common factor.*

For example, as 4 is the greatest common divisor of 12 and 16, so $a^2 - b^2$ is the highest common factor of $a^2 - b^2$ and $a^4 - b^4$.

Although it is not customary to speak of a quantity as a factor of itself, we do so in connection with the highest common factor if one quantity is contained in another.

The letters H.C.F. stand for highest common factor.

The subject is of little importance in elementary algebra, since in reducing fractions to lowest terms we usually cancel the factors one at a time. It is therefore treated only briefly in this work.

123. Finding the Highest Common Factor. In all cases needed in elementary algebra the highest common factor may be found by separating the expressions into factors.

1. Find the H.C.F. of $6x^2 + 7x - 3$ and $4x^2 - 4x - 15$.

$$6x^2 + 7x - 3 = (2x + 3)(3x - 1).$$
$$4x^2 - 4x - 15 = (2x + 3)(2x - 5).$$

Since $2x + 3$ is the only common factor, it is the H.C.F.

2. Find the H.C.F. of $ax^4 - ay^4$ and $a^2x^4 - 2a^2x^2y^2 + a^2y^4$.

$$\begin{aligned} ax^4 - ay^4 &= a(x^4 - y^4) \\ &= a(x^2 + y^2)(x^2 - y^2) \\ &= a(x^2 + y^2)(x + y)(x - y). \\ a^2x^4 - 2a^2x^2y^2 + a^2y^4 &= a^2(x^4 - 2x^2y^2 + y^4) \\ &= a^2(x^2 - y^2)^2 \\ &= a^2(x + y)(x - y)(x + y)(x - y). \end{aligned}$$

The factors common to both expressions are a, $x + y$, and $x - y$. Therefore $a(x + y)(x - y)$ is the H.C.F.

Exercise 96. Highest Common Factor

Examples 1 to 12, oral — Examples 13 to 28, written

1. Name any common factor of 20 and 30.

2. Name any common factor of ab^2c^3 and a^3b^2c.

3. What is the H.C.F. of ab^2 and a^2b? of ab^3 and a^3b?

Find the H.C.F. of the following:

4. $a^3b^3,\ a^2b^2$.

5. $a^4b^2,\ a^2b^4$.

6. $a^2b^2c^2,\ a^3b^3$.

7. $x^4y^4z^4,\ y^3z^5$

8. $x^5y^4z^3,\ x^4y^5$.

9. $x^6y^4z^2,\ x^4y^6$.

10. $a(b+c),\ a^2$.

11. $a^2(b-c),\ a^3b^3$.

12. $a^2-b^2,\ b-a$.

13. Find a common factor of $2a^2-3ab+b^2,\ 3a^2-2ab-b^2$.

Find the H.C.F. of the following:

14. $2a^2+ab-b^2,\ 3a^2+4ab+b^2$.

15. $3a^2+4ab+b^2,\ 2a^2+5ab+3b^2$.

16. $5a^2+34a-7,\ 6a^2+44a+14$.

17. $6a^2+7a+2,\ 8a^2-14a-9$.

18. $a^4-a^3b,\ ab^4-b^6,\ b^2-a^2$.

19. $a^7+a^6b,\ ab^3+b^4,\ a^2+b^2+2ab$.

20. $a^2b^2+ab^3,\ a^4b+a^3b^2,\ a^4(ab+2b^2)+a^3b^3$.

21. $a(a-2b)+b^2,\ b^2-a^2,\ a^3-b^3,\ a^4-b^4$.

22. $a^2+b(b+2a),\ a^4-b^4,\ a^2b^2+a^3(2b+a)$.

23. $x^4-2x^2y^2+y^4,\ x^2+y^2+2xy,\ (x+y)^3$.

24. $x^3+6xy^2+5x^2y,\ 2x^3y+5x^2y^2+2xy^3$.

25. $2x^4y^2+7x^3y^3-9x^2y^4,\ 2x^3y^3+11x^2y^4+9xy^5$.

26. $x^2+3x+2,\ x^3+6x^2+11x+6$.

Reduce the following to lowest terms by canceling the H.C.F. from both numerator and denominator:

27. $\dfrac{x^4-y^4}{x^4-2x^2y^2+y^4}$.

28. $\dfrac{x^6-y^6}{(x+y)(x^2+xy+y^2)}$.

124. Multiple. An algebraic expression which contains another algebraic expression as a factor is called a *multiple* of that expression.

For example, 6 is a multiple of 2 and 3, and $a^2 - b^2$ is a multiple of $a + b$ and $a - b$.

125. Common Multiple. An algebraic expression which is a multiple of two or more expressions is called a *common multiple* of those expressions.

For example, 12 is a common multiple of 2, 3, 4, and 6. Likewise $a^4 - b^4$ is a common multiple of $a + b$ and $a - b$, of $a^2 - b^2$ and $a^2 + b^2$.

126. Lowest Common Multiple. The multiple of lowest degree that *contains* two or more algebraic expressions as factors is called their *lowest common multiple.*

The abbreviation for lowest common multiple is L.C.M.

For example, 12 is a common multiple of 2 and 3, but 6 is their L.C.M. Similarly, $a^4 - b^4$ is a common multiple of $a + b$ and $a - b$, but $a^2 - b^2$ is their L.C.M.

The L.C.M. is not necessarily the least common multiple for all values of the letters. Thus, if $a = 4$ and $b = 2$, the L.C.M. of $a + b$ and $a - b$, which is $a^2 - b^2$, reduces to $16 - 4 = 12$. The L.C.M. of $a + b$ and $a - b$, or 6 and 2, is, however, 6.

In speaking of the L.C.M., a quantity is considered as a multiple of itself. Thus, $a^2 - b^2$ is the L.C.M. of $a^2 - b^2$ and $a - b$. It is also the L.C.M. of $b^2 - a^2$ and $a - b$, the factor -1 not being considered because it does not affect the *degree.*

127. Finding the Lowest Common Multiple. Since the L.C.M. contains each quantity, it must contain all the factors of each, and since it is to be of lowest possible degree, it must contain no unnecessary factors. Therefore

The L.C.M. of two or more expressions must contain all the different factors of the expressions, each factor being taken the greatest number of times that it occurs in any of the given expressions.

For example, the L.C.M. of ab^2 and a^2b is a^2b^2, a and b each being taken twice as a factor.

1. Find the L.C.M. of $15\,a^3b$, $20\,ab^5$, and $30\,a^4b^4$.

$$\begin{aligned} 15\,a^3b &= 3 \cdot 5 \cdot a^3b, \\ 20\,ab^5 &= 2 \cdot 2 \cdot 5 \cdot ab^5, \\ 30\,a^4b^4 &= 2 \cdot 3 \cdot 5 \cdot a^4b^4. \end{aligned}$$

The L.C.M. must contain each quantity, and hence it must contain all the factors of each quantity.

Since it is to be of the lowest degree, it can contain no factor of higher degree than occurs in any one of the quantities.

Therefore the $\quad \text{L.C.M.} = 2 \cdot 2 \cdot 3 \cdot 5 \cdot a^4b^5$

$$= 60\,a^4b^5.$$

2. Find the L.C.M. of $6\,a^2 + 11\,ab + 3\,b^2$ and $4\,a^2 - 4\,ab - 15\,b^2$.

$$\begin{aligned} 6\,a^2 + 11\,ab + 3\,b^2 &= (2\,a + 3\,b)(3\,a + b). \\ 4\,a^2 - 4\,ab - 15\,b^2 &= (2\,a + 3\,b)(2\,a - 5\,b). \\ \therefore \text{ the L.C.M.} &= (2\,a + 3\,b)(3\,a + b)(2\,a - 5\,b). \end{aligned}$$

Exercise 97. The Lowest Common Multiple

Examples 1 to 3, oral — Examples 4 to 15, written

1. Find the L.C.M. of ab^2 and a^2b; of $5\,a^4b^4$ and $3\,a^2b^2$.
2. Find the L.C.M. of $a + b$ and $a - b$; of $a + b$ and $a^2 - b^2$.
3. Find the L.C.M. of $5\,abc$, $6\,a^2b^2c^2$, and $10\,a^3b^3c^3$.

Find the L.C.M. of the following:

4. $34\,a^6b^2c^5$, $51\,a^5b^5c^5$.
5. $38\,x^4y^4z^4$, $57\,x^3y^4z^5$.
6. $58\,m^2n^4x^9$, $87\,m^9n^4x^2$.
7. $x^3 + 1$, $x^2 - 1$.
8. $x^3 + x$, $x^4 - 1$.
9. $x^2 + 2\,x$, $x^2 - 4$.
10. $a^2 + 5\,ab + 6\,b^2$, $a^2 + 8\,ab + 15\,b^2$.
11. $m^2 + 16\,mn + 63\,n^2$, $m^2 - 2\,mn - 63\,n^2$.
12. $x^2 - y^2$, $(y - x)^2$, $(x + y)^2$.
13. $x^2 + 9\,x + 14$, $x^2 + 7\,x + 10$, $x^2 - 7\,x - 18$.
14. $y - x$, $x^2 - y^2$, $x^3 - y^3$, $x^4 - y^4$.
15. $a^3 + 3\,a^2b + 3\,ab^2 + b^3$, $a^3 + b^3$, $a^2 - b^2$.

CHAPTER XI

FRACTIONS

128. Algebraic Fraction. An expression in the form of $\frac{a}{b}$, in which either a or b is an algebraic expression, is called an *algebraic fraction.*

For example, $\frac{2}{x}$, $\frac{x+y}{x-y}$, and $\frac{x+y}{2}$ are algebraic fractions.

Since we cannot divide by zero, b cannot be zero.

Because of its relation to factoring we have already (§ 101) introduced some simple work in fractions.

129. Terms of a Fraction. In the fraction $\frac{a}{b}$, a is called the *numerator,* b is called the *denominator,* and the two together are called the *terms* of the fraction.

The numerator represents the dividend and the denominator represents the divisor of an indicated division, just as in arithmetic.

130. Reduction of Fractions. As already stated (§ 101), we reduce algebraic fractions to fractions having lower terms just as in arithmetic, by dividing both terms by their common factors. Similarly, we reduce to fractions having higher terms by multiplying both terms by the same quantity.

Just as $\frac{10}{15}=\frac{2}{3}$, so $\frac{a+b}{a^2-b^2}=\frac{a+b}{(a+b)(a-b)}=\frac{1}{a-b}$.

Similarly, $\frac{a}{b}=\frac{ax}{bx}$, and $\frac{a}{x}=\frac{a\cdot ax}{x\cdot ax}=\frac{a^2x}{ax^2}$.

Multiplying or dividing both numerator and denominator of a fraction by the same expression does not change the value of the fraction.

131. Reduction of Fractions to Lowest Terms. A fraction is reduced to a fraction in lowest terms when the latter fraction has its numerator and denominator prime to each other.

Briefly, this is called the reduction of a fraction to lowest terms.

To reduce a fraction to lowest terms, divide both numerator and denominator by their common factors.

We might divide by the H.C.F., but since we find the H.C.F. by factoring it is easier to divide by the factors as we discover them.

When a line is drawn through the factors by which both terms of the fraction are divided, the factors are said to be *canceled.*

We have already considered this subject in § 101 and in connection with various cases in factoring. We now review it as a preparation for the further study of fractions.

1. Reduce $\frac{24x^3y^4z}{32x^2y^5z^3}$ to lowest terms.

Dividing both terms by 8, x^2, y^4, and z we have $\frac{3x}{4yz^2}$.

2. Reduce $\frac{a^3-b^3}{a^4-b^4}$ to lowest terms.

$$\frac{a^3-b^3}{a^4-b^4}=\frac{\cancel{(a-b)}(a^2+ab+b^2)}{\cancel{(a-b)}(a+b)(a^2+b^2)}=\frac{a^2+ab+b^2}{(a+b)(a^2+b^2)}.$$

3. Reduce $\frac{x^3+5x^2+8x+6}{x^3-2x-4}$ to lowest terms.

Since it is difficult to factor numerator and denominator we resort to a simple and convenient device.

Since a factor of each of two quantities is a factor of their difference (just as a is a factor of ab and ac, and hence of $ab-ac$), we subtract the denominator from the numerator. The result is $5x^2+10x+10$.

Hence if there is any common factor it is contained in $5x^2+10x+10$, or $5(x^2+2x+2)$. Evidently 5 is not a common factor. Trying x^2+2x+2 we find, by actual division, that it is a common factor. Therefore we have

$$\frac{x^3+5x^2+8x+6}{x^3-2x-4}=\frac{(x+3)\cancel{(x^2+2x+2)}}{(x-2)\cancel{(x^2+2x+2)}}=\frac{x+3}{x-2}.$$

Students should be warned against canceling *terms;* only *factors* can be canceled.

Exercise 98. Reduction of Fractions to Lowest Terms

Examples 1 to 3, oral — Examples 4 to 27, written

1. Reduce to lowest terms: $\frac{a}{ab}, \frac{a^3}{a^2b}, \frac{a^2}{a^3b}, \frac{a^2b^2}{a^3b}.$

2. Reduce to lowest terms: $\frac{ax}{ay}, \frac{a^2x}{ay^2}, \frac{2a^2x}{4ay^2}, \frac{5a^2x^2}{10a^3y^3}.$

3. Reduce to lowest terms: $\frac{abc}{a^2bc}, \frac{abc}{a^2b^2c^2}, \frac{abc^2}{a^2b^2c^2}, \frac{a^2b^2c^2}{a^3b^3c^3}.$

Reduce the following to lowest terms:

4. $\frac{8a^2b^3c^4}{12a^4b^3c^2}.$ **6.** $\frac{24a^{10}b^3}{32a^3b^{10}}.$ **8.** $\frac{35x^4y^4z}{49x^5y^6z}.$ **10.** $\frac{56x^ny}{72x^ny^2}.$

5. $\frac{15a^4b^6c^3}{20a^3b^5c^{10}}.$ **7.** $\frac{36a^{10}b^{10}}{48a^{12}b^{15}}.$ **9.** $\frac{50x^3y^9z^2}{75x^{10}y^{10}z}.$ **11.** $\frac{63x^ny^2}{81x^ny^{10}}.$

12. Reduce $\frac{(a+b)^2}{a^2-b^2}$ to lowest terms, and check the result by letting $a = 3$, $b = 2$.

Reduce the following to lowest terms:

13. $\frac{a^2-b^2}{(a-b)^2}.$ **15.** $\frac{a^3-1}{a^2-1}.$ **17.** $\frac{p^2+q^2}{p^6+q^6}.$ **19.** $\frac{x^3-y^3}{x^6-y^6}.$

14. $\frac{a^2-b^2}{a^3-b^3}.$ **16.** $\frac{m^4-n^4}{m^2+n^2}.$ **18.** $\frac{x^4-16}{x^6-64}.$ **20.** $\frac{x^2-y^2}{x^5-y^5}.$

21. Reduce $\frac{1+4a+4a^2}{1-4a^2}$ to lowest terms, and check the result by letting $a = 2$.

Reduce the following to lowest terms:

22. $\frac{a^3+a^2b}{a^2+2ab+b^2}.$ **25.** $\frac{x^3+2x^2+2x+1}{x^3+3x^2+3x+2}.$

23. $\frac{x^4-y^4}{x^2-2xy+y^2}.$ **26.** $\frac{x^3-8x-8}{x^3+x^2-10x-12}.$

24. $\frac{4x^2-7x+3}{5x^2-3x-2}.$ **27.** $\frac{a^3-6a^2-6a+1}{a^3-2a^2-34a+5}.$

132. The Sign of a Fraction. The plus or minus sign before a fraction is called the *sign of the fraction.*

If there is no sign expressed the plus sign is understood as usual.

133. Changing Signs in the Terms. Since, from the law of signs in division (§ 73),

$$\frac{a}{b}=\frac{-a}{-b}=-\frac{-a}{b}=-\frac{a}{-b},$$

we see that

The value of a fraction is not altered by changing the signs of the numerator and denominator; by changing the signs of the fraction and numerator; or by changing the signs of the fraction and denominator.

It must be remembered that to change the sign of the numerator means that we must change the sign of *every term* of the numerator, and similarly for the denominator.

134. Changing Signs of Factors. Since we may change the signs of any even number of factors without changing their product (§ 120), or of two expressions without changing their quotient (§ 73), therefore

The signs of an even number of factors in the numerator, or in the denominator, may be changed without altering the value of the fraction.

If the signs of an odd number of factors are changed in the numerator, or in the denominator, the sign of the fraction must be changed.

Reduce to lowest terms $\dfrac{(a+b^2)(a-b)}{(b+a^2)(b-a)}$.

$$\frac{(a+b^2)(a-b)}{(b+a^2)(b-a)}=-\frac{(a+b^2)\cancel{(a-b)}}{(b+a^2)\cancel{(a-b)}}=-\frac{a+b^2}{b+a^2}.$$

Pupils should again be warned emphatically against endeavoring to cancel terms instead of factors. Thus it is absurd to say that $\dfrac{2+\cancel{3}}{4+\cancel{3}}=\dfrac{2}{4}=\dfrac{1}{2}$, for the value is evidently $\frac{5}{7}$, which does not equal $\frac{1}{2}$.

Exercise 99. Reduction of Fractions to Lowest Terms

Examples 1 to 4, oral — Examples 5 to 27, written

1. Reduce to lowest terms: $\dfrac{a-b}{2a-2b}$; $\dfrac{a-b}{2b-2a}$.

2. Reduce to lowest terms: $\dfrac{x-y^2}{y^2-x}$; $\dfrac{m^2-n^2}{n^2-m^2}$; $\dfrac{m-n}{n^2-m^2}$.

Reduce the following to lowest terms:

3. $\dfrac{a^2-b^2}{(b-a)^2}$.

4. $\dfrac{m^4-n^4}{n^2+m^2}$.

5. $\dfrac{a^3-1}{1-a^2}$.

6. $\dfrac{a^2-b^2}{b^3-a^3}$.

7. $\dfrac{p^2+q^2}{q^4-p^4}$.

8. $\dfrac{x^4-y^4}{y^6-x^6}$.

9. $\dfrac{y^3-x^3}{x^6-y^6}$.

10. $\dfrac{y^2-x^2}{x^5-y^5}$.

11. Reduce $\dfrac{(b-c)^2-a^2}{(a+b)^2-c^2}$ to lowest terms; also $\dfrac{a^2-(b-c)^2}{c^2-(a+b)^2}$.

Reduce the following to lowest terms:

12. $\dfrac{x^2+5x+6}{x^2+6x+8}$.

13. $\dfrac{a^2+a-6}{8-2a-a^2}$.

14. $\dfrac{a^2-9}{12-a-a^2}$.

15. $\dfrac{x^2+2x-35}{40-3x-x^2}$.

16. $\dfrac{6x^2-5x-4}{6x^2+x-12}$.

17. $\dfrac{6x^2-5x-4}{12-x-6x^2}$.

18. $\dfrac{2x^2+13x+21}{2x^2+10x+12}$.

19. $\dfrac{3x^2-2x-21}{3x^2+19x+28}$.

20. $\dfrac{3x^3+5x^2+4x+2}{3x^3-x^2-2}$.

21. $\dfrac{4a^3+13a^2+4a+3}{4a^3+11a^2-2a+3}$.

22. $\dfrac{4a^3+13a^2+4a+3}{4a^3-11a^2-2a-3}$.

23. $\dfrac{5a^3-4a^2+a-2}{5a^3+6a^2+3a+2}$.

24. $\dfrac{3a^3+2a^2-a+14}{3a^3+4a^2-3a+2}$.

25. $\dfrac{a^3-6a^2+11a-6}{7a-a^3-6}$.

26. $\dfrac{2a^3+14a^2+33a+21}{2a^3+17a^2+27a+12}$.

27. $\dfrac{3a^3+22a^2+47a+28}{23a+21-3a^3-a^2}$.

135. Equivalent Fractions. When two fractions are such that either may be obtained from the other by multiplying or dividing both of its terms by the same expression they are called *equivalent fractions.*

For example, $\frac{6}{12}$ and $\frac{1}{2}$ are equivalent fractions. Likewise $\frac{a+b}{a-b}$ and $\frac{(a+b)^2}{a^2-b^2}$ are equivalent fractions.

136. Common Denominator. Two or more fractions that have the same denominator are said to have a *common denominator.*

137. Lowest Common Denominator. If the common denominator of several fractions is of the lowest degree possible, the fractions are said to have the *lowest common denominator.*

For example, $\frac{(a+b)^2}{a^2-b^2}$ and $\frac{a^2+3ab+2b^2}{a^2-b^2}$ have a common denominator, a^2-b^2. But since they can be reduced to the equivalent fractions $\frac{a+b}{a-b}$ and $\frac{a+2b}{a-b}$, $a-b$ is their *lowest* common denominator (L.C.D.).

138. Finding the Lowest Common Denominator. When several fractions are given in their lowest terms, we reduce any one of them to an equivalent fraction having any required denominator by multiplying both terms by the same expression. Therefore any common denominator of several fractions must be a common multiple of the given denominators. Therefore

The lowest common denominator of several fractions is the lowest common multiple of their denominators.

1. Reduce $\frac{a}{bc}$ and $\frac{b}{cd}$ to equivalent fractions having the lowest common denominator.

The L.C.D. must contain all the factors of bc and cd, and be of the lowest possible degree.

It must therefore contain b, c, and d, and is bcd.

Evidently if we multiply both terms of $\frac{a}{bc}$ by d, and both terms of $\frac{b}{cd}$ by b, each fraction will have the L.C.D. bcd.

Therefore the result is $\frac{ad}{bcd}$ and $\frac{b^2}{bcd}$.

2. Reduce $\frac{2a^2b}{3x^2y}$, $\frac{3ab^2}{2xy^2}$, and $\frac{ab}{xy}$ to equivalent fractions having the lowest common denominator.

The L.C.D. must contain all the factors of $3x^2y$, $2xy^2$, and xy, and be of the lowest possible degree.

It must therefore contain 3, x^2, 2, and y^2, and therefore is $6x^2y^2$.

Dividing $6x^2y^2$ by the respective denominators we have $2y$, $3x$, $6xy$, which are therefore the factors of $6x^2y^2$ not found in these denominators.

Multiplying both terms of the fractions by these factors, in this order, we have for the required fractions

$$\frac{4a^2by}{6x^2y^2},\quad \frac{9ab^2x}{6x^2y^2},\quad \frac{6abxy}{6x^2y^2}.$$

Fractions should be expressed in lowest terms before beginning to find the L.C.D.

3. Reduce $\frac{x-y}{x^2+3xy+2y^2}$ and $\frac{x}{2x^2+3xy+y^2}$ to equivalent fractions having the lowest common denominator.

The L.C.D. must contain the two denominators

$$x^2+3xy+2y^2=(x+y)(x+2y)$$

and

$$2x^2+3xy+y^2=(x+y)(2x+y).$$

Therefore the $\quad$ L.C.D. $=(x+y)(x+2y)(2x+y)$.

Dividing, the multiplying factors are $2x+y$ and $x+2y$, respectively.

Therefore

$$\frac{x-y}{(x+y)(x+2y)}=\frac{(x-y)(2x+y)}{(x+y)(x+2y)(2x+y)}$$

$$\frac{x}{(x+y)(2x+y)}=\frac{x(x+2y)}{(x+y)(x+2y)(2x+y)}.$$

4. Reduce $\frac{x-y}{x+y}$ and $\frac{x+y}{x-y}$ to equivalent fractions having the lowest common denominator. Then let $x=4$ and $y=2$ and interpret the result.

The L.C.D. is $(x+y)(x-y)$.

The resulting fractions are $\frac{(x-y)^2}{(x+y)(x-y)}$ and $\frac{(x+y)^2}{(x+y)(x-y)}$.

Substituting the values we have $\frac{4}{12}$ and $\frac{36}{12}$. That is, the *lowest* common denominator is not always the *least* common denominator for all values of the letters.

Exercise 100. Reducing Fractions to the Lowest Common Denominator

Examples 1 to 11, oral — Examples 12 to 27, written

1. Express with L.C.D.: $\frac{a}{b^2}, \frac{1}{ab}; \frac{1}{a^2}, \frac{1}{ab}; \frac{a}{b}, \frac{b}{c}$.

2. Express with L.C.D.: $\frac{1}{a-b}, \frac{1}{a+b}; \frac{1}{a+b}, \frac{1}{(a+b)^2}$.

Express the following with lowest common denominator:

3. $\frac{1}{a}, \frac{1}{b}$.

4. $\frac{1}{a^2}, \frac{1}{a^3}$.

5. $\frac{1}{ab^2}, \frac{1}{a^2b}$.

6. $\frac{a+b}{a-b}, \frac{1}{a+b}$.

7. $\frac{a-b}{a+b}, \frac{1}{a-b}$.

8. $\frac{ab}{xyz}, \frac{a^2b^2}{x^2y^2z^2}$.

9. $\frac{p}{3q^2}, \frac{q}{2p^2}$.

10. $\frac{3}{a+b}, \frac{2}{a-b}$.

11. $\frac{a}{x(y+z)}, \frac{b}{y(y+z)}$.

12. $\frac{a}{x+y}, \frac{b}{x-y}$.

13. $\frac{x}{x+3}, \frac{x}{x-3}$.

14. $\frac{a+b}{a-b}, \frac{a-b}{a^2+ab+b^2}$.

15. $\frac{p^2-1}{p^2+1}, \frac{p^2+1}{p^2-1}$.

16. $\frac{a-b}{a^3+b^3}, \frac{a+b}{a^2-ab+b^2}$.

17. $\frac{a^2+b^2}{a^2-b^2}, \frac{a+b}{a-b}$.

18. $\frac{1}{x^2-y^2}, \frac{1}{x^3-y^3}$.

19. $\frac{1}{a^3-b^3}, \frac{1}{b^2-a^2}$.

20. $\frac{1}{x+y}, \frac{2}{x-y}, \frac{3}{x^2-y^2}$.

21. $\frac{a^2}{a-b}, \frac{a}{a^2-b^2}, \frac{a-b}{a+b}$.

22. $\frac{1}{a-b}, \frac{1}{b-c}, \frac{1}{c-a}$.

23. $\frac{a+3}{a-3}, \frac{a^2+9}{a^2-9}, \frac{a^3+27}{a^3-27}$.

24. $\frac{x}{y+z}, \frac{y}{z+x}, \frac{z}{x+y}$.

25. $\frac{2}{a}, \frac{3}{b}, \frac{4}{c}, \frac{9}{a+b+c}$.

26. $\frac{1}{(a-b)(b-c)}, \frac{1}{(c-b)(a-c)}$.

27. $\frac{1}{(x-y)(y-z)}, \frac{1}{(x-z)(z-y)}$.

139. Addition and Subtraction of Fractions. In algebra, as in arithmetic, fractions are added by first expressing them with the lowest common denominator.

Thus we might add $\frac{3}{4}$ and $\frac{5}{8}$ by reducing them to $\frac{24}{32}$ and $\frac{20}{32}$, but it saves labor to reduce them to $\frac{6}{8}$ and $\frac{5}{8}$, the sum being $\frac{11}{8}$, or $1\frac{3}{8}$.

So we might add $\frac{a}{b}$ and $\frac{a^2}{b^2}$ by reducing them to $\frac{ab^2}{b^3}$ and $\frac{a^2b}{b^3}$ and then adding, but it saves labor to use the lowest common denominator and to add $\frac{ab}{b^2}$ and $\frac{a^2}{b^2}$, the sum being $\frac{ab+a^2}{b^2}$, or $\frac{a(a+b)}{b^2}$.

Find the algebraic sum of $\frac{a-b}{a+b}+\frac{a+b}{a-b}-\frac{a^2+b^2}{a^2-b^2}$.

Since to subtract a quantity is the same as to add its negative, we may treat of addition and subtraction at the same time.

The denominators are $a+b$, $a-b$, and $(a+b)(a-b)$.

Therefore the L.C.D. is $(a+b)(a-b)$, or a^2-b^2.

Reducing to equivalent fractions with this denominator, we have

$$\frac{(a-b)^2}{a^2-b^2}+\frac{(a+b)^2}{a^2-b^2}-\frac{a^2+b^2}{a^2-b^2}=\frac{a^2-2ab+b^2+a^2+2ab+b^2-a^2-b^2}{a^2-b^2}$$
$$=\frac{a^2+b^2}{a^2-b^2}.$$

It should be noticed that the fraction line is a symbol of aggregation, so that when the third fraction is subtracted the signs of both a^2 and b^2 become negative.

Check. Let $a=2$, $b=1$. We then have $\frac{1}{3}+\frac{3}{1}-\frac{5}{3}=\frac{1+9-5}{3}=\frac{5}{3}$. The result, $\frac{a^2+b^2}{a^2-b^2}$, also equals $\frac{5}{3}$.

In the above example the work may conveniently be arranged as follows:

The L.C.D is $(a+b)(a-b)$.

The respective multipliers are $a-b$, $a+b$, and 1.

$$\begin{array}{rlll}
(a-b)(a-b) = & a^2-2ab & +\,b^2 & = \text{1st numerator.}\\
(a+b)(a+b) = & a^2+2ab & +\,b^2 & = \text{2d numerator.}\\
-(a^2+b^2) = & \underline{-a^2} & \underline{-\,b^2} & = \text{3d numerator.}\\
 & a^2 & +\,b^2 & = \text{sum of numerators.}
\end{array}$$

Therefore the sum of the fractions $=\frac{a^2+b^2}{a^2-b^2}$.

140. Finding the Algebraic Sum of Fractions. Therefore, to find the algebraic sum of several fractions we proceed as follows:

If the fractions have the same denominators, write the algebraic sum of the numerators over the common denominator.

If the fractions have different denominators express them as equivalent fractions having the lowest common denominator, and write the algebraic sum of the new numerators over the common denominator.

In either case, reduce the resulting fraction to lowest terms.

Exercise 101. Addition and Subtraction of Fractions

Examples 1 to 6, oral — Examples 7 to 61, written

1. Add $\frac{a}{b}, \frac{c}{b}$; $\frac{a}{x^2}, \frac{b}{x^2}$; also $\frac{a}{m}, -\frac{b}{m}$.

2. Add $\frac{a+b}{x}, \frac{a-b}{x}$; $\frac{a-b}{m}, \frac{a-b}{m}$; also $\frac{a+b}{c}, \frac{b+a}{c}$.

Add the following:

3. $\frac{a^2-b^2}{x+y}, \frac{a^2+b^2}{x+y}$.

4. $\frac{a^2-25}{x-y}, \frac{a^2+25}{x-y}$.

5. $\frac{a}{2}, \frac{b}{2}, -\frac{c}{2}, -\frac{b}{2}$.

6. $\frac{a-b}{4}, \frac{b-c}{4}, \frac{c}{4}$.

7. Add $\frac{a+b}{a-b}$ and $\frac{2ab}{a^2-b^2}$. Check the result by letting $a=3$, $b=2$.

8. $\frac{a+b}{4} - \frac{a+2b}{6}$.

9. $\frac{x-y}{2} + \frac{y-x}{12}$.

10. $\frac{2mn}{5} - \frac{m^2-2mn+n^2}{15}$.

11. $\frac{b-a}{5} + \frac{a-b}{3}$.

12. $\frac{x+y}{4} - \frac{x-y}{8}$.

13. $\frac{m^2-n^2}{5} + \frac{m^2+2mn+n^2}{10}$.

14. $\frac{a}{3}+\frac{3a}{4}-\frac{5a}{12}$.

15. $\frac{x}{5}-\frac{2x}{3}+\frac{7x}{15}$.

16. $\frac{2a}{7}-\frac{a}{3}+\frac{5a}{21}$.

17. $\frac{4m}{5}-\frac{7m}{10}+\frac{4m}{15}$.

18. $\frac{3}{a}+\frac{4}{a^2}-\frac{5}{a^3}$.

19. $\frac{3}{a^3}-\frac{2}{3a^2}+\frac{1}{a}$.

20. $\frac{2}{3y}+\frac{3}{4y^2}-\frac{5}{12}$.

21. $\frac{5}{4x}-\frac{7}{5x}+\frac{1}{10x^2}$.

22. $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$.

23. $\frac{x}{y}-\frac{y}{z}+\frac{z}{x}$.

24. $\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}$.

25. $\frac{x}{yz}-\frac{y}{zx}-\frac{z}{xy}$.

26. $\frac{a}{m^2}+\frac{b}{mn}-\frac{c}{n^2}$.

27. $\frac{m^2}{n^2}-\frac{2}{mn}+\frac{n^2}{m^2}$.

28. $\frac{x}{ab}+\frac{y}{bc}+\frac{z}{ca}$.

29. $\frac{a+b}{a-b}+\frac{a-b}{a+b}$.

30. $\frac{a+b}{a-b}-\frac{a+b}{a^3-b^3}$.

31. $\frac{x}{x-y}-\frac{y}{x^3-y^3}$.

32. $\frac{a^2}{(a-1)^2}-\frac{a}{a-1}$.

33. $\frac{3}{a-3}+\frac{4}{a+3}$.

34. $\frac{5}{a-2}-\frac{3}{a+2}$.

35. $\frac{x}{x^2-y^2}+\frac{y}{x^3-y^3}$.

36. $\frac{x}{x^2-y^2}-\frac{y}{x^3+y^3}$.

37. $\frac{a-b}{(a+b)^2}+\frac{a+b}{(a-b)^2}$.

38. $\frac{x+5}{(x-5)^2}-\frac{x-5}{(x+5)^2}$.

39. $\frac{a+b}{a^3-b^3}+\frac{a-b}{a^3+b^3}$.

40. $\frac{x+7}{x-3}+\frac{x+3}{x-7}$.

41. $\frac{x^2+1}{x^2-4}-\frac{x^2-1}{x^2+4}$.

42. $\frac{a+5}{a+6}+\frac{a+6}{a+7}$.

43. $\frac{a^2+2ab+b^2}{a-b}+\frac{(a-b)^2}{a+b}$.

44. $\frac{a^2-2ab+b^2}{a+b}+\frac{(a+b)^2}{a-b}$.

45. $\frac{a^2-2ab+b^2}{a+b}+\frac{a^2+b^2}{a-b}$.

46. $\frac{a^2-2ab+b^2}{(a+b)^2}+\frac{(a+b)^2}{a^2-2ab+b^2}$.

47. $\frac{a+b}{a^2+ab+b^2}+\frac{a-b}{a^2-ab+b^2}$.

48. $\frac{2a-b}{4a^2-2ab+b^2}+\frac{2a+b}{4a^2+2ab+b^2}$.

49. $\frac{p^2+pq+q^2}{p^3-q^3}-\frac{p^2-pq+q^2}{p^3+q^3}$.

50. $\frac{3x}{x+4}+\frac{4x}{x+3}+\frac{5}{x^2+7x+12}$.

51. $\frac{x^2+2xy+y^2}{x^2-2xy+y^2}-\frac{x^2-2xy+y^2}{x^2+2xy+y^2}-\frac{8xy^3}{(x^2-y^2)^2}$.

52. $\frac{m+2}{m^2+m}+\frac{m-2}{m}-\frac{m}{m^2+2m+1}$.

53. $\frac{a+b}{x-y}+\frac{a-b}{x+y}+\frac{a^2-b^2}{x^2-y^2}$.

54. $\frac{a^2+ab+b^2}{a^2-ab+b^2}-\frac{a^2-ab+b^2}{a^2+ab+b^2}$.

55. $\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}$.

56. $\frac{a+b}{a-b}+\frac{b+c}{b-c}+\frac{c+a}{c-a}$.

57. $\frac{a}{b^2-bc}+\frac{b}{c^2-ca}+\frac{c}{a^2-ab}$.

58. $\frac{x^2-yz}{(x+y)(z+x)}+\frac{y^2-zx}{(y+z)(x+y)}+\frac{z^2-xy}{(z+x)(y+z)}$.

59. $\frac{1}{a^2+7a+12}+\frac{2}{a^2+8a+15}+\frac{3}{a+3}$.

60. $\frac{1}{a^2-(b-c)^2}-\frac{1}{b^2-(c-a)^2}-\frac{1}{c^2-(a-b)^2}$.

61. $\frac{x+a}{x^2-(b+c)x+bc}+\frac{x+b}{x^2-(c+a)x+ca}+\frac{x+c}{x^2-(a+b)x+ab}$.

141. Special Suggestions in Addition. The problems already solved are typical of most of the problems that will be met in the addition of fractions. A few other forms are occasionally found, and three of these types will now be considered.

1. Add $\frac{a}{b}$, $-\frac{a}{3b-2}$, and $\frac{2b+1}{4-9b^2}$.

Since $4-9b^2=(2+3b)(2-3b)$, we change the sign of the second fraction and of its denominator (§ 133).

$$\frac{a}{b}+\frac{a}{2-3b}+\frac{2b+1}{4-9b^2}=\frac{a(4-9b^2)}{b(4-9b^2)}+\frac{ab(2+3b)}{b(4-9b^2)}+\frac{b(2b+1)}{b(4-9b^2)}$$
$$=\frac{4a-6ab^2+2ab+2b^2+b}{b(4-9b^2)}.$$

2. Find the value of $\frac{1}{x^2+3x-10}-\frac{1}{6-5x+x^2}$.

$x^2+3x-10=(x+5)(x-2)$.

$6-5x+x^2=(3-x)(2-x)$, or $(x-3)(x-2)$, by § 120.

We therefore have

$$\frac{1}{(x+5)(x-2)}-\frac{1}{(x-3)(x-2)}=\frac{-8}{(x+5)(x-2)(x-3)}.$$

3. Find the value of

$$\frac{1}{(x-y)(y-z)}+\frac{1}{(z-y)(x-z)}+\frac{1}{(z-x)(y-x)}.$$

We see that $y-z$ is the same as $z-y$ except for its signs, and similarly for $x-y$ and $y-x$, and for $x-z$ and $z-x$.

But in the second fraction $(z-y)(x-z)=(y-z)(z-x)$, by § 120.

In the third fraction $(z-x)(y-x)=-(z-x)(x-y)$.

We may therefore write the fractions thus:

$$\frac{1}{(x-y)(y-z)}+\frac{1}{(y-z)(z-x)}-\frac{1}{(z-x)(x-y)}.$$

These equal

$$\frac{z-x}{(x-y)(y-z)(z-x)}+\frac{x-y}{(x-y)(y-z)(z-x)}-\frac{y-z}{(x-y)(y-z)(z-x)}$$
$$=\frac{2z-2y}{(x-y)(y-z)(z-x)}=\frac{-2}{(x-y)(z-x)}.$$

Exercise 102. Addition and Subtraction of Fractions

Examples 1 to 8, oral — Examples 9 to 17, written

1. What is the sum of $\frac{1}{a}$ and $\frac{1}{-a}$?

2. Add $\frac{1}{a}$, $\frac{1}{-a}$, $\frac{-1}{a}$, and $\frac{-1}{-a}$.

Add or subtract as indicated:

3. $\frac{2}{a}+\frac{1}{-a}$.

4. $\frac{5}{a^2}-\frac{1}{-a^2}$.

5. $\frac{1}{xy}-\frac{2}{-xy}$.

6. $\frac{3}{mn}-\frac{4}{-mn}$.

7. $\frac{1}{a-b}+\frac{1}{b-a}$.

8. $\frac{3}{x-4}+\frac{1}{4-x}$.

9. Find the value of $\frac{1}{(x-2)(x-3)}+\frac{1}{(x-1)(2-x)}$, and check the result by letting $x = 4$.

Add or subtract as indicated:

10. $\frac{1}{(a-b)(a+b)}+\frac{2}{(b-a)(a+b)}+\frac{3}{b^2-a^2}$.

11. $\frac{4a}{(a-5)(a-b)}-\frac{3a}{(5-a)(a-b)}$.

12. $\frac{2}{a^2-10a+21}+\frac{2}{a-7}-\frac{3}{3-a}$.

13. $\frac{5}{x^2-14x+45}-\frac{3}{x-5}+\frac{2}{9-x}$.

14. $\frac{1}{(x-1)(x-2)}+\frac{2}{(x-2)(3-x)}-\frac{3}{(x-3)(1-x)}$.

15. $\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-c)(b-a)}+\frac{1}{c(c-a)(c-b)}$.

16. $\frac{3}{(p-q)(q-r)}-\frac{4}{(r-q)(p-r)}+\frac{2}{(r-p)(p-q)}$.

17. $\frac{a}{(x-y)(y-z)}+\frac{b}{(z-x)(y-x)}+\frac{c}{(y-z)(x-z)}$.

142. Mixed Expression. An algebraic expression that is the sum of an integer and a fraction is called a *mixed expression.*

For example, $2\frac{1}{3}$ is a mixed number, and $2a + \frac{1}{3}b$ is a mixed expression. In arithmetic the plus sign is omitted between the integer and fraction in a mixed number, $2\frac{1}{3}$ meaning $2 + \frac{1}{3}$. In algebra $a\frac{b}{c}$ means $a \times \frac{b}{c}$, and hence the plus sign must be used in a mixed expression, as in the case of $a + \frac{b}{c}$.

143. Reduction of a Fraction to a Mixed Expression. If the degree of the numerator of a fraction equals or exceeds that of the denominator, in a given letter, the fraction may be reduced to an integer or a mixed expression by dividing the numerator by the denominator.

Just as we reduce $\frac{25}{15}$ to $1\frac{2}{3}$ by dividing, so we may reduce $\frac{a^2+b^2}{a+b}$ to $a - b + \frac{2b^2}{a+b}$. We cannot, however, reduce $\frac{x^2+y^2}{a+b}$, although the numerator is of the second degree and the denominator is of the first degree, because different letters are involved.

1. Reduce $\frac{4x^2 - 6xy + 3y^2}{2x - y}$ to a mixed expression.

Dividing,
$$\frac{4x^2 - 6xy + 3y^2}{2x - y} = 2x - 2y + \frac{y^2}{2x - y}.$$

2. Reduce $\frac{a^3 + 3a^2b - 6ab^2 + 2b^3}{a^2 + 2ab + b^2}$ to a mixed expression.

Dividing, $\frac{a^3 + 3a^2b - 6ab^2 + 2b^3}{a^2 + 2ab + b^2} = a + b$, with a remainder of $-9ab^2 + b^3$. We may therefore write the result

$$a + b + \frac{-9ab^2 + b^3}{a^2 + 2ab + b^2},$$

or (§ 133)

$$a + b - \frac{9ab^2 - b^3}{a^2 + 2ab + b^2},$$

the latter being the more common form.

The work may be checked by substituting any value for the letters that will not make the denominator zero.

Exercise 103. Reduction of a Fraction to a Mixed Expression

Examples 1 to 11, oral — Examples 12 to 26, written

1. Reduce to an integer: $\frac{10}{2}$; $\frac{10a^2}{2}$; $\frac{10a^2}{2a}$; $\frac{a^2b}{ab}$; $\frac{(a+b)^2}{a+b}$.

2. Reduce to an integer: $\frac{a^2-b^2}{a-b}$; $\frac{a^2-b^2}{a+b}$; $\frac{a^3+b^3}{a+b}$; $\frac{a^3-b^3}{a-b}$.

3. Reduce to a mixed expression: $\frac{x+y}{x}$; $\frac{x-y}{x}$; $\frac{ab+c}{a}$.

Reduce to integral or mixed expressions:

4. $\frac{ax-ay}{a}$.

5. $\frac{x^4-y^4}{x^2+y^2}$.

6. $\frac{abx+aby}{ab}$.

7. $\frac{abx+1}{ab}$.

8. $\frac{mn^2+3}{mn}$.

9. $\frac{abc+c}{abc}$.

10. $\frac{m^6-n^6}{m^3-n^3}$.

11. $\frac{a+b}{a-b}$.

12. Reduce $\frac{a^2+3ab+4b^2}{a+b}$ to a mixed expression, and check the result by letting $a=2$ and $b=3$.

Reduce the following to integral or mixed expressions:

13. $\frac{x^2+3xy+y^2}{x+y}$.

14. $\frac{x^2-3xy-y^2}{x-y}$.

15. $\frac{x^2+15x+56}{x+7}$.

16. $\frac{x^2-15x+56}{x-8}$.

17. $\frac{a^2b^2+3ab+2}{ab+2}$.

18. $\frac{p^2+4pq+q^2}{p+q}$.

19. $\frac{m^2-13mn+4n^2}{m-n}$.

20. $\frac{m^3+3m^2n+3mn^2+n^3}{m^2+n^2+2mn}$.

21. $\frac{m^3+3m^2n+4mn^2+5n^3}{m(m+2n)+n^2}$.

22. $\frac{a^3+a^2b+ab^2+b^3}{a+b}$.

23. $\frac{a^3+b^3+c^3-3abc}{a+b+c}$.

24. $\frac{x^3+3x^2y-xy^2-y^3}{x+y}$.

25. $\frac{x^4+x^2y^2+y^4}{x^2+xy+y^2}$.

26. $\frac{x^4+4x^2y^2+16y^4}{x^2+2xy+4y^2}$.

144. Reduction of a Mixed Expression to a Fraction. Since the value of an expression is not changed if it is both multiplied and divided by the same expression, then $a = \frac{ab}{b}$, and $a + \frac{a}{b} = \frac{ab + a}{b}$. Therefore

To reduce a mixed expression to a fraction, multiply the integral part by the denominator, to the product add the numerator, and under the result write the denominator.

This is simply the same as adding two fractions. For example, $a + \frac{c}{d} = \frac{ad}{d} + \frac{c}{d} = \frac{ad + c}{d}$.

Reduce $a + 3b - \frac{a^2 - 2ab + 4b^2}{a - 3b}$ to a fraction.

Reducing $a + 3b$ to a fraction with $a - 3b$ for its denominator, we have

$$\frac{a^2 - 9b^2}{a - 3b} - \frac{a^2 - 2ab + 4b^2}{a - 3b} = \frac{a^2 - 9b^2 - a^2 + 2ab - 4b^2}{a - 3b} = \frac{2ab - 13b^2}{a - 3b}.$$

Exercise 104. Reduction of a Mixed Expression to a Fraction

Examples 1 to 3, oral — Examples 4 to 9, written

1. Reduce to an improper fraction: $2\frac{3}{8}$; $4\frac{5}{8}$; $9\frac{7}{8}$.

2. Reduce to a fraction: $a + \frac{b}{c}$; $a - \frac{b}{c}$; $a + b + \frac{c}{d}$.

3. Reduce to a fraction: $a + \frac{ac}{b + c}$; $a - \frac{ac}{b + c}$; $a + b + \frac{c}{a - b}$.

Reduce the following to fractions:

4. $a + b + \frac{a^2 + b^2}{a - b}$.

5. $x + 2y + \frac{4y^2}{x - 2y}$.

6. $a^2 + ab + b^2 + \frac{2b^3}{a - b}$.

7. $\frac{a^2 + b^2}{a - b} + b + 4a$.

8. $a^2 - ab + b^2 - \frac{b^3}{a + b}$.

9. $\frac{x^2 + 2xy - y^2}{x + y} + x + y$.

145. Multiplication of Fractions. We multiply fractions in algebra in the same manner as in arithmetic. Just as $5\times\frac{2}{3}=\frac{10}{3}$, so $a\cdot\frac{b}{c}=\frac{ab}{c}$. Therefore

To multiply a fraction by an integral expression, multiply the numerator, writing the product over the denominator.

Similarly, just as $\frac{2}{3}$ of $\frac{4}{5}$ equals $\frac{2\times 4}{3\times 5}$, or $\frac{8}{15}$, so $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$. Therefore

To multiply a fraction by a fraction, multiply the numerators together for a new numerator, and the denominators together for a new denominator.

Cancel factors common to any numerator and any denominator before multiplying.

1. Find the product of $\frac{x^2y}{2m^2n}\times 4\times\frac{5mn^3}{6xy^4}\times\frac{3y^2}{5m}$.

Indicating the multiplication, we have

$$\frac{3\cdot 4\cdot 5\cdot mn^3x^2y^3}{2\cdot 5\cdot 6\cdot m^3nxy^4}.$$

Canceling common factors, we have $\frac{n^2x}{m^2y}$.

2. Find the product of $\frac{x^2-x-6}{x^2+x-6}\cdot\frac{x^2+2x-8}{x^2-2x-8}\cdot\frac{x^2-x-12}{x^2+x-12}$.

Factoring, for ease in canceling, and then canceling common factors, we have

$$\frac{\cancel{(x+2)}\cancel{(x-3)}}{\cancel{(x-2)}\cancel{(x+3)}}\cdot\frac{\cancel{(x-2)}\cancel{(x+4)}}{\cancel{(x+2)}\cancel{(x-4)}}\cdot\frac{\cancel{(x+3)}\cancel{(x-4)}}{\cancel{(x-3)}\cancel{(x+4)}}=1.$$

3. Simplify $\left(x-1-\frac{4}{x-1}\right)\left(x^3+3x^2+8x+24+\frac{72}{x-3}\right)$.

Reducing to fractional forms,

$$\frac{x^2-2x-3}{x-1}\cdot\frac{x^4-x^2}{x-3}=\frac{\cancel{(x-3)}(x+1)}{\cancel{x-1}}\cdot\frac{x^2(x+1)\cancel{(x-1)}}{\cancel{x-3}}=x^2(x+1)^2.$$

Check. Let $x=2$. Then $(2-1-4)(8+12+16+24-72)=-3\cdot -12=36$, and $4\cdot 9=36$.

Exercise 105. Multiplication of Fractions

Examples 1 to 12, oral — Examples 13 to 46, written

1. $b \times \frac{a}{b}$.
2. $4 \times \frac{a}{4b}$.
3. $4b \times \frac{a}{4b}$.
4. $ab \times \frac{a}{b^3}$.
5. $mn \cdot \frac{a}{mn}$.
6. $p^2q \cdot \frac{1}{pq}$.
7. $x^2y^2 \cdot \frac{1}{xy}$.
8. $x^3y \cdot \frac{1}{xy^3}$.
9. $\frac{a}{b} \cdot \frac{p}{q}$.
10. $\frac{m}{n} \cdot \frac{n}{m}$.
11. $\frac{m}{n^2} \cdot \frac{n}{m^2}$.
12. $\frac{ab}{xy} \cdot \frac{ax}{by}$.

13. Find the product of $\frac{2ax^2}{3by^2} \cdot \frac{3cz^2}{4ax^2} \cdot \frac{4by^2}{5cz^2}$. Check the result by letting $a = b = c = 1$, and $x = y = z = 2$.

14. $\frac{ab^2}{c^2d} \cdot \frac{bc^2}{d^2a} \cdot \frac{cd^2}{a^2b}$.

15. $\frac{a^2x}{b^2y} \cdot \frac{b^2y}{c^2z} \cdot \frac{c^2z}{a^2x}$.

16. $\frac{m^3n}{pq^2} \cdot \frac{mn^2}{p^2q} \cdot \frac{p^4q^4}{m^4n^4}$.

17. $\frac{(2a)^2}{b} \cdot \frac{(3b)^2}{c} \cdot \frac{(4c)^2}{a}$.

18. $\frac{-a}{b^2} \cdot \frac{-b}{c^2} \cdot \frac{-c}{a^2}$.

19. $\frac{(8ab)^3}{(3pq)^2} \cdot \frac{(6pq)^3}{(4ab)^2} \cdot \frac{abc}{pqr}$.

20. Find the product of $\frac{x^2 - y^2}{x^2 + y^2} \cdot \frac{x + y}{x - y} \cdot \frac{x^4 - y^4}{4}$. Check the result by letting $x = 2$, $y = 1$.

21. $\frac{a^2 - b^2}{a^2 + b^2} \cdot \frac{4a}{a + b}$.

22. $\frac{a^2 - b^2}{a^2 + b^2} \cdot \frac{2a}{3b + 3a}$.

23. $\frac{a - b}{a + b} \cdot \frac{a^2 - b^2}{a^3 - b^3}$.

24. $\frac{x - 7}{x + 3} \cdot \frac{9 - x^2}{49 - x^2}$.

25. $\frac{x^2 - 1}{x^2 + 1} \cdot \frac{1 - x^4}{1 + x^4}$.

26. $\frac{4x^3 + 8x^2}{x - 3} \cdot \frac{9x - x^3}{2 + x}$.

27. $\frac{(a + b)^2}{a - b} \cdot \frac{a^2 - b^2}{a^2 + b^2}$.

28. $\frac{(m - n)^2}{m + n} \cdot \frac{m^3 + n^3}{n - m}$.

29. $\frac{x^2 + xy}{x^2 - 2xy + y^2} \cdot \frac{y^2 - x^2}{4x}$.

30. $\frac{4x}{3x - 2} \cdot \frac{4 - 9x^2}{8x}$.

31. $\dfrac{a(a-b)}{a^2+2ab+b^2}\cdot\dfrac{b(b+a)}{b^2-2ba+a^2}\cdot\dfrac{a^2-b^2}{ab}$.

32. $\dfrac{x^3+y^3}{x^3-y^3}\cdot\dfrac{x^2+y^2}{x^2-y^2}\cdot\dfrac{x+y}{x-y}\cdot\dfrac{y^2-x^2}{y^2+x^2}$.

33. $\dfrac{m^3-n^3}{m^2-9}\cdot\dfrac{3-m}{m+n}\cdot\dfrac{3+m}{m^2-mn+n^2}$.

34. $\dfrac{p^3-q^3}{p^3+q^3}\cdot\dfrac{p+q}{p-q}\cdot\dfrac{p^2-pq+q^2}{p^2+pq+q^2}$.

35. $\dfrac{a^2+2ab+b^2}{a-b}\cdot\dfrac{a^2-2ab+b^2}{a+b}\cdot\dfrac{a-b}{a+b}$.

36. $\dfrac{x^2-y^2}{p-q}\cdot\dfrac{q^2-p^2}{y-x}\cdot\dfrac{3}{p+q}\cdot\dfrac{3}{x-y}$.

37. $\dfrac{a+b-c}{x}\cdot\dfrac{a-b+c}{y}\cdot\dfrac{x+y-z}{a}\cdot\dfrac{x-y+z}{b}$.

38. $\left(\dfrac{a}{m}-\dfrac{m}{a}+\dfrac{n}{b}-\dfrac{b}{n}\right)\left(\dfrac{a}{m}-\dfrac{m}{a}-\dfrac{n}{b}+\dfrac{b}{n}\right)$.

39. $\left(\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a}\right)\left(\dfrac{b}{a}+\dfrac{c}{b}+\dfrac{a}{c}\right)$.

40. $\dfrac{x^2-5x+6}{x^2-9x+20}\cdot\dfrac{x^2-11x+28}{x^2-8x+12}\cdot\dfrac{x^2-11x+30}{x^2-10x+21}$.

41. $\dfrac{x^2+3x+2}{x^2+8x+15}\cdot\dfrac{x^2+7x+12}{x^2-5x-6}\cdot\dfrac{x^2-x-30}{x^2+6x+8}$.

42. $\dfrac{x^2-1}{x^2+5x+6}\cdot\dfrac{x^2-4}{x^2-2x-3}\cdot\dfrac{x^2-9}{x^2-3x+2}$.

43. $\dfrac{x^2-x-2}{x^2+8x+15}\cdot\dfrac{x^2-x-12}{x^2+x-42}\cdot\dfrac{x^2-x-30}{x^2-7x-8}\cdot\dfrac{x^2-x-56}{x^2-6x+8}$.

44. $\dfrac{6x^2+5x+1}{10x^2-9x+2}\cdot\dfrac{8x^2+2x-3}{5+8x-4x^2}\cdot\dfrac{10x^2-29x+10}{12x^2+13x+3}$.

45. $\dfrac{4a^2-25}{12a^2+11a+2}\cdot\dfrac{9a^2-4}{8a^2+18a-5}\cdot\dfrac{16a^2-1}{6a^2-19a+10}$.

46. $\dfrac{15a^2-41a+14}{10a^2-19a+6}\cdot\dfrac{6a^2-23a+21}{8a^2-10a-3}\cdot\dfrac{12a^2-25a-7}{20a^2-3a-2}$.

146. Reciprocal. If the product of two quantities is 1, each is called the *reciprocal* of the other.

Since $\frac{a}{b} \cdot \frac{b}{a} = 1$, the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$, and conversely.

147. Division of Fractions. Consider the following cases:

\$2 ÷ \$3 = $\frac{2}{3}$, the denomination "dollars" disappearing in the quotient.

2 ft. ÷ 3 ft. = $\frac{2}{3}$, the denomination "feet" disappearing.

$\frac{2}{5} \div \frac{3}{5} = \frac{2}{3}$, the denomination "fifths" disappearing.

To divide $\frac{a}{b}$ by $\frac{c}{d}$ is the same as to divide $\frac{ad}{bd}$ by $\frac{bc}{bd}$ (§ 130).

$\frac{ad}{bd} \div \frac{bc}{bd} = \frac{ad}{bc}$, the denomination "*bd*ths" disappearing.

But $\frac{ad}{bc} = \frac{a}{b} \cdot \frac{d}{c}$. Therefore we get the same result in dividing $\frac{a}{b}$ by $\frac{c}{d}$ that we get by multiplying $\frac{a}{b}$ by $\frac{d}{c}$. Therefore

To divide by a fraction, multiply by its reciprocal.

1\. Divide $\frac{x^2 - x - 6}{x^2 + 3x - 4}$ by $\frac{x^2 + x - 12}{x^2 + x - 2}$.

Multiplying by the reciprocal of the divisor,

$$\frac{x^2 - x - 6}{x^2 + 3x - 4} \cdot \frac{x^2 + x - 2}{x^2 + x - 12} = \frac{(x+2)\cancel{(x-3)}}{(x+4)\cancel{(x-1)}} \cdot \frac{\cancel{(x-1)}(x+2)}{\cancel{(x-3)}(x+4)} = \frac{(x+2)^2}{(x+4)^2}.$$

2\. Simplify $\frac{4x^2 - 1}{6x^2 + 13x + 6} \div \frac{2x^2 + 3x - 2}{2x^2 - 5x - 12} \div \frac{2x^2 - 7x - 4}{3x^2 + 8x + 4}$

Multiplying by the reciprocals of the divisors,

$$\frac{4x^2 - 1}{6x^2 + 13x + 6} \cdot \frac{2x^2 - 5x - 12}{2x^2 + 3x - 2} \cdot \frac{3x^2 + 8x + 4}{2x^2 - 7x - 4}$$

$$= \frac{\cancel{(2x+1)}\cancel{(2x-1)}}{\cancel{(3x+2)}\cancel{(2x+3)}} \cdot \frac{\cancel{(2x+3)}\cancel{(x-4)}}{\cancel{(2x-1)}\cancel{(x+2)}} \cdot \frac{\cancel{(x+2)}\cancel{(3x+2)}}{\cancel{(2x+1)}\cancel{(x-4)}} = 1.$$

Check. Substituting 1 for x,

$$\frac{3}{25} \div \frac{3}{-15} \div \frac{-9}{15} = \frac{3 \cdot -15 \cdot 15}{25 \cdot 3 \cdot -9} = 1.$$

Exercise 106. Division of Fractions

Examples 1 to 11, oral — Examples 12 to 24, written

1. Divide $\frac{a}{2}$ by $\frac{a}{4}$; $\frac{a}{4}$ by $\frac{a}{2}$; $\frac{2}{a}$ by $\frac{4}{a}$; $\frac{4}{a}$ by $\frac{2}{a}$; $\frac{4}{a}$ by $\frac{a}{2}$.

2. Divide $\frac{a+b}{x}$ by $\frac{a-b}{x}$; $\frac{a+b}{2x}$ by $\frac{a-b}{2x}$; $\frac{a+b}{x+y}$ by $\frac{a-b}{x+y}$.

3. $\frac{ab}{xy} \div \frac{cd}{xy}$.

4. $\frac{ab}{xy} \div \frac{bc}{xy}$.

5. $\frac{abc}{xy} \div \frac{bcd}{xy}$.

6. $\frac{2x}{y} \div \frac{3x}{y}$.

7. $\frac{2x}{y} \div \frac{3w}{y}$.

8. $\frac{2x}{y} \div \frac{3xy}{y^2}$.

9. $\frac{m}{n} \div \frac{n}{m}$.

10. $\frac{a}{b} \div \frac{m}{n}$.

11. $\frac{ab}{cd} \div \frac{cd}{ab}$.

12. Divide $\frac{15a^2b}{16c^2d}$ by $\frac{3ab^2}{4cd^2}$, and check the result by letting $a = 2$, $b = 3$, $c = 4$, $d = 5$.

13. $\frac{25abc}{32xyz} \div \frac{15a^2b^2c^2}{28x^2y^2z^2}$.

14. $\frac{17x^2yz}{24m^3n^2} \div \frac{51xy^2z^3}{75m^4n^5}$.

15. $\frac{35a^4b^5c^6}{27x^4y^5z^6} \div \frac{28a^3b^2c}{81x^2y^3z^4}$.

16. $\frac{25m^3n^4z}{33a^4b^6c^6} \div \frac{35mnz^4}{44a^3b^3c^3}$.

17. $\frac{a^3+b^3}{a-b} \div \frac{a^3-b^3}{a+b}$.

18. $\frac{a^3+b^3}{a-b} \div \frac{a+b}{a^3-b^3}$.

19. $\frac{a^4-b^4}{a^4+b^4} \div \frac{a^2-b^2}{a^2+b^2}$.

20. $\frac{m^2+n^2}{m^2-n^2} \div \frac{2m^2+2n^2}{m-n}$.

21. Divide $\frac{p^2-4q^2}{p^2+4pq}$ by $\frac{p^2-2pq}{pq+4q^2}$, and check the result by letting $p = 5$, $q = 2$.

22. $\frac{a^2+b^2-c^2+2ab}{c^2-a^2-b^2+2ab} \div \frac{a+b+c}{b+c-a}$.

23. $\frac{a^3+b^3+3ab(a+b)}{a^3-b^3-3ab(a-b)} \div \frac{a(a+2b)+b^2}{a(a-2b)+b^2}$.

24. $\left(\frac{1+m}{1-m} - \frac{1-n}{1+n}\right) \div \left[1 + \frac{(1+m)(1-n)}{(1-m)(1+n)}\right]$.

148. Complex Fraction. A fraction that has one or more fractions in either or both of its terms is called a *complex fraction.*

A complex fraction is merely an indicated case of the division of fractions. Thus $\dfrac{1+\dfrac{1}{x-1}}{1-\dfrac{1}{x+1}}$ simply indicates that $1+\dfrac{1}{x-1}$ is to be divided by $1-\dfrac{1}{x+1}$.

Reducing both terms of the complex fraction to fractional forms and simplifying, we have a fraction in which we may multiply both terms by $(x-1)(x+1)$, or which we can treat as a case of division, thus:

$$\frac{1+\dfrac{1}{x-1}}{1-\dfrac{1}{x+1}}=\frac{\dfrac{x}{x-1}}{\dfrac{x}{x+1}}=\frac{x(x+1)}{x(x-1)}=\frac{x+1}{x-1}.$$

1. Simplify $\dfrac{\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}}{\dfrac{x^2+y^2}{x^2-y^2}}$.

Multiply both numerator and denominator of the complex fraction by x^2-y^2, which is the L.C.D. of the simple fractions. This reduces the complex fraction to

$$\frac{(x+y)^2+(x-y)^2}{x^2+y^2}=\frac{x^2+2xy+y^2+x^2-2xy+y^2}{x^2+y^2}$$
$$=\frac{2x^2+2y^2}{x^2+y^2}=2.$$

2. Simplify $\dfrac{1}{x+\dfrac{1}{x+\dfrac{1}{x}}}$.

Multiply both terms of the last complex fraction by x. This gives $\dfrac{x}{x^2+1}$, the complex fraction then reducing to $\dfrac{1}{x+\dfrac{x}{x^2+1}}$. Multiplying both terms of this fraction by x^2+1, we have $\dfrac{x^2+1}{x^3+2x}$.

A complex fraction of this form is also called a *continued fraction.*

Exercise 107. Complex Fractions

Examples 1 to 5, oral — Examples 6 to 17, written

1. Simplify $\dfrac{1+\frac{1}{2}}{1-\frac{1}{2}};\ \dfrac{1+\frac{1}{4}}{1-\frac{1}{4}};\ \dfrac{1-\frac{1}{5}}{1+\frac{1}{5}};\ \dfrac{1-\frac{1}{6}}{1+\frac{1}{6}};\ \dfrac{\frac{1}{4}}{1+\frac{1}{4}}.$

2. Simplify $\dfrac{\frac{1}{2}}{1+\frac{1}{2}};\ \dfrac{\frac{2}{3}}{1-\frac{1}{3}};\ \dfrac{\frac{2}{3}}{1+\frac{1}{3}};\ \dfrac{\frac{3}{4}}{1-\frac{3}{4}};\ \dfrac{\frac{3}{4}}{1+\frac{3}{4}}.$

Simplify the following:

3. $\dfrac{\dfrac{a+b}{a}}{\dfrac{a-b}{b}}.$

4. $\dfrac{a+\dfrac{a}{b}}{a-\dfrac{a}{b}}.$

5. $\dfrac{x+\dfrac{1}{y}}{x-\dfrac{1}{y}}.$

6. $\dfrac{\dfrac{1+a}{a}}{1-\dfrac{1}{a^2}}.$

7. $\dfrac{\dfrac{a+b}{a-b}}{\dfrac{a-b}{a+b}}.$

8. $\dfrac{\dfrac{x-y}{x+y}}{\dfrac{x^2-y^2}{x^2+y^2}}.$

9. $\dfrac{\dfrac{a^3}{b^3}+1}{1+\dfrac{a}{b}}.$

10. $\dfrac{16-\dfrac{a^4}{b^4}}{\dfrac{a}{b}+2}.$

11. $\dfrac{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}{\dfrac{a-b}{a+b}+\dfrac{a+b}{a-b}}.$

12. $\dfrac{x-1+\dfrac{y}{x-y}}{x-2+\dfrac{y}{x+y}}.$

13. $\dfrac{a}{1+\dfrac{b}{c+\dfrac{1}{d}}}.$

14. $\dfrac{\dfrac{p+q}{p}-\dfrac{p-q}{q}}{\dfrac{p+q}{q}+\dfrac{p-q}{p}}.$

15. $\dfrac{a-b+\dfrac{c}{a+b}}{a+b-\dfrac{c}{a-b}}.$

16. $\dfrac{x}{y+\dfrac{1}{y+\dfrac{1}{x}}}.$

17. Simplify $\dfrac{m+\dfrac{mn}{m-n}}{m-\dfrac{mn}{m-n}}$, and check the result.

CHAPTER XII

SIMPLE FRACTIONAL EQUATIONS

149. Clearing of Fractions. Multiplying both members of an equation by such a quantity as shall leave no fractions in the equation is called *clearing the equation of fractions.*

If $\frac{1}{2}x = 5,$

by multiplying by 2, $x = 10,$ an equation cleared of fractions.

If $\frac{x}{a} + \frac{x}{b} = ab,$

by multiplying by ab, $bx + ax = a^2b^2$, an equation cleared of fractions.

To clear an equation of fractions, multiply both members of the equation by the lowest common multiple of the denominators.

If a fraction is preceded by a minus sign, change the sign of every term in the numerator when the denominator is removed.

After clearing of fractions, solve the equation in the usual manner.

1. Solve $\frac{x}{2} + \frac{3}{a} - 4 = \frac{1}{a} - \frac{1}{4}.$

Multiplying by $4a$, the L.C.M. of the denominators,

$$2ax + 12 - 16a = 4 - a.$$

Subtracting $12 - 16a$, $2ax = -12 + 16a + 4 - a.$

Combining, $2ax = 15a - 8.$

Dividing by $2a$, $x = \frac{15a - 8}{2a}.$

2. Solve $\frac{x+1}{x-1} = \frac{x-1}{x-2}.$

Multiplying by the L.C.M., $x^2 - x - 2 = x^2 - 2x + 1.$

Therefore $x = 3.$

Exercise 108. Fractional Equations

Examples 1 to 4, oral — Examples 5 to 30, written

1. Clear of fractions: $\frac{x}{a}=b$; $\frac{x}{2a}=\frac{a}{2}$; $\frac{x}{3}+\frac{1}{4}=1$.

2. Solve $\frac{x}{3}+1=2$; $\frac{x}{4}-1=2$; $\frac{x}{5}-2=2$.

3. Solve $\frac{2x}{3}=4$; $\frac{3x}{4}=6$; $\frac{4x}{5}=8$; $\frac{5x}{7}=10$.

4. Solve $\frac{x+1}{2}=3$; $\frac{x-1}{2}=5$; $\frac{x+1}{3}=6$; $\frac{x-1}{5}=2$.

Solve the following equations:

5. $\frac{x+1}{x-1}=\frac{5}{3}$.

6. $\frac{x-1}{x+1}=\frac{3}{4}$.

7. $\frac{x+2}{x-2}=1\frac{2}{3}$.

8. $\frac{x-2}{x+2}=\frac{3}{7}$.

9. $\frac{x+3}{x-5}=9$.

10. $\frac{x-3}{x-7}=3$.

11. $\frac{x+1}{x-1}=\frac{2x-5}{2x-7}$.

12. $\frac{x+2}{x-3}=\frac{2x-5}{2x-10}$.

13. $\frac{x-5}{x+6}=\frac{3x-17}{3x-6}$.

14. $\frac{2x-1}{2x+1}=1-\frac{2}{3x}$.

15. $\frac{2x+3}{2x-5}=\frac{4x-3}{4x-11}$.

16. $\frac{5x+1}{5x-1}=\frac{2x+3}{2(x+1)}$.

17. $\frac{4x-1}{3}-\frac{3}{4}=\frac{x-4}{6}+\frac{3x+5}{4}$.

18. $\frac{2x}{5}+\frac{9}{10}-\frac{x+5}{5}=\frac{7x}{25}-\frac{x+3}{20}-\frac{1}{25}$.

19. $\frac{4x}{15}-\frac{9}{10}-\frac{16x-81}{24}=\frac{2x-3}{15}-\frac{4x-9}{20}+\frac{9}{40}$.

20. $\frac{2x}{7}-\frac{3}{8}-\frac{5x-30}{16}=\frac{4x+7}{14}-\frac{x-4}{8}-\frac{13}{16}$.

21. Half of a certain number exceeds one eighth of the number by 6. Find the number.

Let $x =$ the number.

Then $\frac{x}{2} - \frac{x}{8} =$ the excess of its half over its eighth,

$= 6.$

Multiplying by 8, $4x - x = 48.$

Combining, $3x = 48.$

Dividing by 3, $x = 16.$

Check. $\frac{16}{2} - \frac{16}{8} = 8 - 2 = 6.$

22. One third of a certain number exceeds one fifth of the number by 2. Find the number.

23. The sum of the seventh and ninth parts of a certain number is 16. Find the number.

24. The sum of one half, one third, and one fourth of a certain number is 13. Find the number.

25. The sum of one half and three fourths of a certain number is two more than the number. Find the number.

26. There are two consecutive numbers, x and $x + 1$, such that half of the first plus one third of the second equals 7. Find the numbers.

27. There are three consecutive numbers, such that one seventh of the first, plus one fifth of the second, plus one eighth of the third equals half of the first. Find the numbers.

28. If the excess of a certain number over nine is divided by the sum of the number and nine, the quotient is $\frac{1}{4}$. Find the number.

29. One fifth of the excess of a certain number over three equals one sixth of the sum of the number and three. Find the number.

30. One half of a certain number, plus one fourth of the number, plus one eighth of the number, plus one sixteenth of the number equals 120. Find the number.

150. Combining Terms before Clearing. It often simplifies the work to combine certain terms before clearing an equation of fractions.

1. Solve $\frac{x+1}{2a}+\frac{2x}{3a}=\frac{1}{2a}+\frac{x}{3a}+5.$

Here it is apparently advisable to unite the fractions with the same denominators before clearing of fractions.

Subtracting $\frac{1}{2a}$ and $\frac{x}{3a}$,

$$\frac{x+1-1}{2a}+\frac{2x-x}{3a}=5,$$

or

$$\frac{x}{2a}+\frac{x}{3a}=5.$$

Combining, $\frac{5x}{6a}=5.$

Dividing by 5, $\frac{x}{6a}=1.$

Multiplying by $6a$, $x=6a.$

2. Solve $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-5}{x-6}-\frac{x-6}{x-7}.$

Combining the fractions in the first member and those in the second member, we have

$$\frac{(x-1)(x-3)-(x-2)^2}{(x-2)(x-3)}=\frac{(x-5)(x-7)-(x-6)^2}{(x-6)(x-7)}.$$

Simplifying the numerators,

$$\frac{-1}{(x-2)(x-3)}=\frac{-1}{(x-6)(x-7)}.$$

Multiplying by -1, and then clearing of fractions,

$$(x-6)(x-7)=(x-2)(x-3).$$

Expanding, $x^2-13x+42=x^2-5x+6.$

Therefore $8x=36,$

and $x=4\frac{1}{2}.$

We might also reduce to mixed fractions and then solve, thus:

$$1+\frac{1}{x-2}-1-\frac{1}{x-3}=1+\frac{1}{x-6}-1-\frac{1}{x-7}.$$

Exercise 109. Fractional Equations

Examples 1 to 3, oral — Examples 4 to 26, written

1. Solve $x + \frac{3}{2} - \frac{1}{2} = 2$; $x + \frac{4}{5} = 3 - \frac{1}{5}$.

2. Solve $x + \frac{2}{3} - \frac{1}{3} = 2\frac{1}{3}$; $x + \frac{3}{4} + \frac{1}{4} = 4$.

3. Solve $x + \frac{7}{8} - \frac{3}{8} = 5\frac{1}{2}$; $x + \frac{5}{8} + \frac{3}{8} = 17$.

Solve the following equations:

4. $\dfrac{3x-5}{5x-5} + \dfrac{5x-1}{7x-7} + \dfrac{x-4}{x-1} = 2.$

5. $\dfrac{8x+2}{x-2} - \dfrac{2x-1}{3x-6} + \dfrac{3x+2}{5x-10} + 5 = 15.$

6. $\dfrac{3x-1}{2x-6} + \dfrac{5x-7}{3x-9} + \dfrac{7x+1}{4x-12} + 14 = 25$

7. $\dfrac{1}{5+7x} - \dfrac{9-11x}{5-7x} + \dfrac{3x}{5+7x} = \dfrac{14(2x-3)^2}{25-49x^2}.$

8. $\dfrac{2}{x+1} - \dfrac{2}{x-1} = \dfrac{1}{x(x+1)}.$

9. $\dfrac{5x}{x-1} - \dfrac{2}{x-1} - \dfrac{2}{x+1} + \dfrac{7}{x+1} = \dfrac{5x^2+4}{x^2-1}.$

10. $\dfrac{6}{x+2} - \dfrac{x}{x-2} - \dfrac{2}{2-x} = \dfrac{x^2}{4-x^2}.$

11. $\dfrac{2}{x-1} + \dfrac{5}{2(1-x)} - \dfrac{8}{3(x-1)} - \dfrac{5}{18} = \dfrac{x}{1-x}.$

12. $\dfrac{2x}{2x-1} - \dfrac{2x-1}{2x+1} - \dfrac{1}{1-2x} = \dfrac{8}{4x^2-1}.$

13. $\dfrac{3x}{2x-2} + \dfrac{5}{6(x-1)} - \dfrac{15x-7}{9(x+1)} = \dfrac{17x-x^2}{6(x^2-1)} + \dfrac{71}{18(x^2-1)}$

14. $\dfrac{4}{x+1} + \dfrac{x}{1-x} - \dfrac{1}{x-1} = \dfrac{x^2-3}{1-x^2}.$

Solve the following equations:

15. $\frac{4}{x+2}+\frac{7}{3+x}-\frac{37}{x(x+5)+6}=0.$

16. $\frac{1}{x-1}-\frac{1}{x-2}=\frac{1}{x-3}-\frac{1}{x-4}.$

17. $\frac{1}{x+1}+\frac{2}{x-1}=\frac{4}{x+1}.$

18. One fifteenth of $6x+7$, divided by x, equals the quotient of $2x-2$ divided by $5x-6$. Find the value of x.

19. One fifteenth of $6x+1$, diminished by the quotient of $2x-4$ divided by $7x-16$, equals one fifth of $2x-1$. Find the value of x.

20. One fourteenth of $11x-13$, diminished by one twenty-eighth of $22x-75$, equals one half of the quotient of $13x+7$ divided by $3x+7$. Find the value of x.

21. One thirty-sixth of $9x+20$ equals one fourth of x added to the quotient of $4x-12$ divided by $5x-4$. Find the value of x.

22. If from twice a certain number we subtract 4, and divide this difference by 5 more than the number, the quotient is $1\frac{1}{15}$. Find the number.

23. If we divide 7 more than a certain number by 7 less than the number the quotient is 15. Find the number.

24. The sum of 5 and a certain number is divided by the sum of 3 and the number. The quotient is $1\frac{1}{6}$. Find the number.

25. The sum of 7 and twice a certain number is divided by 7 less twice the number. The quotient is 1.8. Find the number.

26. If from one tenth of the sum of four times a certain number and three there is subtracted one fifth of twice the number less one, the result is the quotient of the excess of twice the number over five divided by the excess of five times the number over one. Find the number.

151. Monomial and Polynomial Denominators. When an equation contains fractions some of which have monomial denominators, it is usually advisable to clear of these fractions first.

Solve $$x^2+\frac{1}{2}+\frac{9x-7}{6(3x+11)}=\frac{3x^2+2}{3}.$$

Multiplying by 6, $$6x^2+3+\frac{9x-7}{3x+11}=6x^2+4.$$

Subtracting $6x^2+3$, $$\frac{9x-7}{3x+11}=1.$$

Multiplying by $3x+11$, $$9x-7=3x+11,$$

Solving this equation, $$x=3.$$

Exercise 110. Monomial and Polynomial Denominators

Examples 1 and 2, oral — Examples 3 to 8, written

1. Solve $\frac{x}{2}+\frac{x}{3}=5$; $\frac{x}{2}-\frac{x}{3}=10$; $\frac{x}{2}-\frac{x}{4}=7$.

2. Solve $\frac{2x}{3}+\frac{x}{3}=17$; $\frac{3x}{4}-\frac{x}{2}=11$; $\frac{5x}{8}-\frac{x}{8}=12$.

Solve the following equations:

3. $\frac{10x+17}{18}+\frac{4-5x}{9}=\frac{2(6x+1)}{13x-16}.$

4. $\frac{2x}{3}-\frac{2x+4}{3}+\frac{7}{9}=\frac{13-7x}{3(2x+1)}.$

5. $\frac{2x}{5}+\frac{2(1-x)}{7x-6}+\frac{7}{15}=\frac{2x+1}{5}.$

6. $\frac{9x+5}{14}-\frac{7-8x}{2(3x+1)}=\frac{36x+15}{56}+\frac{41}{56}.$

7. $\frac{11x-13}{14}-\frac{22x-75}{28}=\frac{13x+7}{2(3x+7)}.$

8. $\frac{x}{x-1}-\frac{5}{2x-2}=\frac{8}{3x-3}+\frac{x}{1-x}+\frac{5}{18}.$

152. Equations involving Decimal Fractions. In practical measurements we are coming more and more to use decimal fractions, and hence we meet these fractions in equations. They have the advantage that frequently we do not need to clear the equation of fractions.

1. Solve $0.7x - 0.3 = 0.05x + 1$.

Subtracting $0.05x$, $0.65x - 0.3 = 1.$

Adding 0.3, $0.65x = 1.3.$

Dividing by 0.65, $x = 2.$

Check. $0.7 \times 2 - 0.3 = 0.05 \times 2 + 1,$

for $1.4 - 0.3 = 0.1 + 1 = 1.1.$

2. Solve $0.4x + 0.3 - \dfrac{0.4x - 1}{x - 0.2} = \dfrac{0.2x - 0.1}{0.5}.$

Multiplying both terms of the two common fractions by 10,

$$0.4x + 0.3 - \frac{4x - 10}{10x - 2} = \frac{2x - 1}{5}.$$

The second member, $\dfrac{2x - 1}{5}$, evidently equals $0.4x - 0.2$, and hence

$$0.4x + 0.3 - \frac{\cancel{2}(2x - 5)}{\cancel{2}(5x - 1)} = 0.4x - 0.2.$$

Subtracting, $0.5 = \dfrac{2x - 5}{5x - 1}.$

Multiplying by $5x - 1$, $2.5x - 0.5 = 2x - 5.$

Subtracting, $0.5x = -4.5.$

Dividing by 0.5, $x = -9.$

Check. $-3.6 + 0.3 - \dfrac{-3.6 - 1}{-9 - 0.2} = \dfrac{-1.8 - 0.1}{0.5}$, for each member reduces to -3.8.

3. Solve $0.7x + 6.5 = 8 + 0.05x$.

If desired the equation may be cleared of decimal fractions before transposing terms.

Multiplying by 100, $70x + 650 = 800 + 5x.$

Subtracting, $65x = 150.$

Dividing by 65, $x = 2\tfrac{4}{13}.$

Exercise 111. Equations involving Decimal Fractions

Examples 1 to 3, oral — Examples 4 to 23, written

1. Solve $x = 5 + 0.5x$; $x = 7 + 0.5x$; $x = 9 + 0.5x$.

2. Solve $x = 3 + 0.25x$; $x = 6 + 0.25x$; $x = 9 + 0.25x$.

3. Solve $0.2x = 1 - 0.3x$; $0.3x = 1 - 0.2x$; $0.4x = 1 - 0.1x$.

Solve the following equations and check:

4. $0.4x + 5 = 9$.

5. $0.7x + 3 = 11$.

6. $0.9x + 7 = 25$.

7. $0.09x + 7 = 25$.

8. $0.09x + 11 = 10.8$.

9. $0.5x - 6 = 0.2x$.

10. $0.7x - 5 = 0.2x$.

11. $2.5x - 24 = 1.3x$.

12. $3.9x - 4.8 = 2.7x$.

13. $5.3x - 0.68 = 1.9x$.

14. Solve the equation $0.3x + 7.2 = 4.8 - 0.02x$, (1) by clearing of decimal fractions before subtracting any terms from both members; (2) by first subtracting certain terms from both members and then dividing by the decimal coefficient of x.

Solve the following equations and check:

15. $0.7x - 3 + \frac{2}{5}x^2 = 0.3 + 0.4x(1 + x)$.

16. $3.48x - 8 + 0.7x = 1.1x + 5(8 + 0.01x)$.

17. $2.725x + 5(0.1x - 18) = 5(1.635 + 0.1x) - 90$.

18. $5x - 12.42748 = 4(-0.00517 - 0.3x) - 0.0034x$.

19. $2x - 4.03 = 2(2.0518 - 0.0001x) - 0.711x$.

20. $10x + 2 - 3(5x + \frac{1}{2}) = 4x + 0.6 - (x + 0.9)$.

21. Seven increased by 0.75 of a number equals 9.7. What is the number?

22. Four increased by 0.45 of a certain number equals 9.4. What is the number?

23. If to 0.72 of a certain number I add 0.4, the sum is 22. What is the number?

153. Literal Equations. Equations in which some or all of the known quantities are represented by letters are called *literal equations.*

Known quantities are usually represented by the first letters of the alphabet, and unknown numbers by the last letters of the alphabet.

Thus, in the equation $ax + b = cx + d$, x is supposed to represent the unknown quantity unless otherwise stated, and a, b, c, and d to represent known quantities.

We shall see that there are exceptions to this custom, but since they are always apparent from the example they will offer no difficulty.

1. Solve $(a + x)(a - x) = x(3a - x)$.

Here, as usual, unless the contrary is stated, x is supposed to be the unknown quantity.

Simplifying, $a^2 - x^2 = 3ax - x^2$.
Subtracting $-x^2$, $a^2 = 3ax$.
Dividing by $3a$, $\frac{1}{3}a = x$.
Check. $\frac{4}{3}a \cdot \frac{2}{3}a = \frac{1}{3}a \cdot \frac{8}{3}a = \frac{8}{9}a$.

In the solution we may leave the unknown quantity in the second member, as here, or we may transpose, and then change the signs by multiplying both sides by -1, having

$$3ax = a^2,$$
$$x = \tfrac{1}{3}a.$$

2. Solve $ax + b^2 = a^2 - bx$.

Subtracting b^2 and $-bx$, $ax + bx = a^2 - b^2$.
Factoring, $(a + b)x = (a + b)(a - b)$.
Dividing by $a + b$, $x = a - b$.

3. Solve $\dfrac{x}{a} + b = \dfrac{x}{b} + a$.

Multiplying by ab, $bx + ab^2 = ax + a^2b$.
Subtracting ab^2 and ax, $bx - ax = a^2b - ab^2$.
Factoring, $(b - a)x = -ab(b - a)$.
Dividing by $b - a$, $x = -ab$.
Check. $\dfrac{-ab}{a} + b = \dfrac{-ab}{b} + a$,
for $-b + b = -a + a$.

Exercise 112. Literal Equations

Examples 1 to 5, oral — Examples 6 to 29, written

1. If $ax = b$, what is the value of x?

2. If $(a - 1)x = a^2 - 1$, what is the value of x?

3. Solve $4\,abx = 16\,a^2b^2$; $5\,a^2b^2x = 25\,a^3b^3$; $17\,a^2x = 51\,a^4$.

4. Solve $ax = a$; $(a - b)x = a - b$; $(a - b)x = b - a$; $(a - b)x = a^2 - 2\,ab + b^2$.

5. Solve $\dfrac{x}{a} = a$; $\dfrac{x}{a} = b$; $\dfrac{x}{2\,ab} = 1$; $\dfrac{x}{a + b} = a - b$.

Solve the following equations:

6. $2\,ax + 1 = ax + 7$.

7. $2\,ax + a = ax + b$.

8. $2\,x - a^2 = x + a^2$.

9. $x^2 + a^2 = (x + b)^2$.

10. $ax - a^2 = ab - bx$.

11. $a(a - x) = b(b - x)$.

12. $x^2 + ax = (x + a)^2$.

13. $\dfrac{c}{a(x + 1)} = \dfrac{b}{c(1 - x)}$.

14. $\dfrac{a + dx}{cd + d^2x} = \dfrac{a - x}{a - dx}$.

15. $\dfrac{a + bx}{a + b} = \dfrac{c + dx}{c + d}$.

16. $\dfrac{\dfrac{m + n}{x}}{\dfrac{1}{m}} = \dfrac{a}{b}$.

17. $2\,a^2bx - c = 2\,b^2cx - a$.

18. $4\,abcx - bc = 4\,bcx + 1$.

19. $(a + x)(x + b) = x(x - b)$.

20. $(a + x)(a - x) = x(b - x)$.

21. $(x + a)^2 = x(x - b)$.

22. $(x + a)(x - b) = x^2 + abc$.

23. $(4x + b)(x - b) = 4x(x + b)$.

24. $\dfrac{x - a}{x - b} = \left(\dfrac{2\,x - a}{2\,x - b}\right)^2$.

25. $\dfrac{a + b}{x + 2} = \dfrac{b - a}{x - 2}$.

26. $\dfrac{6\,x + a}{4\,x + b} = \dfrac{3\,x - b}{2\,x - a}$.

27. $\dfrac{\dfrac{ax - b}{2}}{\dfrac{a}{b}} = \dfrac{\dfrac{ax + b}{2}}{\dfrac{b}{a}}$.

28. $\dfrac{x + a}{b} - x = b - \dfrac{x - b}{c} + \dfrac{c - bx}{b}$.

29. $\dfrac{x^2 + 2\,ax + a^2}{a + x} + \dfrac{x^2 + a^2 - 2\,ax}{x - a} = \dfrac{x - a}{b} + \dfrac{x - b}{a}$.

154. Letters used in Formulas. In writing formulas such as are used in measuring, physics, and business, it is convenient to use other letters than $a, b, c, \cdots$ for the known numbers, and x, y, z, for the unknown quantities. The initial letter of some word is often used for either a known or an unknown quantity.

For example, we say that $c = 2\pi r$, meaning that the circumference equals 2π times the radius. If we know what r equals and wish to find c, then we think of r as known and c as unknown. If we know what c equals and wish to find r, then c is known and r is unknown.

1. Given $c = 2\pi r$, to find r.

Dividing by 2π, that is, by $2 \times 3\frac{1}{7}$ or 2×3.1416,

$$\frac{c}{2\pi} = r.$$

2. Given $v = \pi r^2 h$, to find h.

Dividing by πr^2, $\frac{v}{\pi r^2} = h.$

3. Given $v = \pi r^2 h$, to find r.

Dividing by πh, $\frac{v}{\pi h} = r^2.$

Extracting the square root $\sqrt{\frac{v}{\pi h}} = r.$

155. Use of Primes and Subscripts. If we wish to use a formula in which two different radii occur, we have found that we may represent one radius by r and the other by r' (read "r prime").

For example, $c = 2\pi r$, and $c' = 2\pi r'$; then $c - c' = 2\pi(r - r')$.
Here we have the advantage of using the initial letter c for two different circumferences and the initial letter r for two different radii.

Sometimes it is convenient to use r' and r'' ("r second") or, preferably, r_1 and r_2 (read "r sub-one" and "r sub-two"). It is also customary, particularly in machine work, to use capital letters in formulas. We read r_1^2 "r sub-one square" or "the square of r sub-one." As already stated, we may distinguish R as "r major," and r as "r minor."

Exercise 113. Formulas as Equations

Examples 1 to 6, oral — Examples 7 to 38, written

1. Solve $a = \pi r^2$ for r.

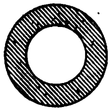
2. Solve $a = \frac{1}{2}bh$ for h; also for b.

3. Solve $v = bwl$ for b; also for w; also for l.

4. Solve $a = \frac{1}{2}h(b + b')$ for h. How will you solve for b?

5. Solve $i = prt$ for p; also for r; also for t.

6. Solve $a = p(1 + rt)$ for p. How will you proceed to solve for r?



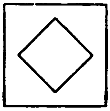
7. The area of a ring is indicated by the formula $a = \pi(r_1 + r_2)(r_1 - r_2)$. Find the value of r_1.

8. In Ex. 7, suppose $a = 286$, $r_1 = 10$, and $\pi = 3\frac{1}{7}$, find the value of r_2.

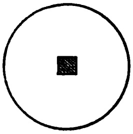
9. In Ex. 7, suppose $a = 506$, $r_2 = 12$, and $\pi = 3\frac{1}{7}$, find the value of r_1.

10. In Ex. 7, suppose $a = 528$ and $r_1 = 2r_2$, find the value of r_1 and r_2.

11. In connecting two wheels by a belt it is often required to know the difference in the two circumferences. This is given by the formula $D = 2\pi(r_1 - r_2)$. Suppose D and r_2 are known, find r_1.



12. If from a square of side s_1 there is cut out a smaller square of side s_2, the area of the part that remains is given by the formula $a = (s_1 + s_2)(s_1 - s_2)$. Find a when $s_1 = 11$, $s_2 = 9$; also find s_2 when a and s_1 are given.



13. If from a circle of radius r there is cut a square of side s the area that remains is given by the formula $a = \pi r^2 - s^2$. If $r = 7$ and $s = 4$, find a. If a and s are known, find r. If a and r are known, find s.

The following formulas will be met by the student in his subsequent work in mathematics or physics. Solve as directed:

14. $s = vt$. Find t.

15. $s = vt$. Find v.

16. $s = \frac{1}{2}gt^2$. Find g.

17. $s = \frac{1}{2}gt^2$. Find t.

18. $d = rt$. Find r.

19. $d = rt$. Find t.

20. $W_1L_2 = W_2L_1$. Find L_1.

21. $W_1L_2 = W_2L_1$. Find W_2.

22. $W = Fs$. Find s.

23. $V_1P_1 = V_2P_2$. Find P_2.

24. $V_1P_1 = V_2P_2$. Find P_1.

25. $D(w_1 - w_2) = w_1$. Find w_2.

26. $F = 32 + \frac{9}{5}C$. Find C.

27. $v_2t = v_1t + n$. Find v_1.

28. $v_2t = v_1t + n$. Find t.

29. $v_2t = v_1t + n$. Find n.

30. $s = \frac{1}{2}n(a + l)$. Solve for n; for a; for l.

31. $s = \dfrac{rl - a}{r - 1}$. Solve for r; for l; for a.

32. $a = \frac{1}{2}h(b + b')$. Solve for h; for b; for b'.

33. $C = \dfrac{en}{R + nr}$. Solve for e; for n; for R; for r.

34. There is a formula in physics, relating to moving bodies, that states that $v = at + \frac{1}{2}gt^2$. Solve for a; for g.

35. The volume of a sphere is expressed by the formula $v = \frac{4}{3}\pi r^3$. Find the value of r^3.

36. The volume of a cylinder of radius r and height h is expressed by the formula $v = \pi r^2h$. Find the value of h; the value of r^2; the value of r.

37. If the side of the square base of a pyramid is s and the height of the pyramid is h, the volume is $v = \frac{1}{3}hs^2$. From this formula find the value of h; the value of s^2; the value of s.

38. The centigrade thermometer is used in scientific work. The number of degrees on the centigrade thermometer can be found from our common (Fahrenheit) thermometer by the formula $C = \frac{5}{9}(F - 32)$, where C is the number of degrees centigrade, and F the number of degrees Fahrenheit. Given $F = 70$, find C. Also find F in terms of C.

Exercise 114. Review Problems

Examples 1 to 4, oral — Examples 5 to 80, written

1. If a square is s feet on a side and a rectangle is n feet longer, how long is the rectangle?

2. If the rectangle in Ex. 1 is m feet narrower than the square, how wide is the rectangle?

3. If $x^2 + 2x - 2 = x^2$, what is the value of x?

4. If $5x^2 + 3x - 21 = 5x^2$, what is the value of x?

5. A rectangle is 3 ft. longer and 2 ft. narrower than a square of the same area. Find the side of the square; the length of the rectangle; the width of the rectangle.

Let $x =$ the number of feet in the side of the square.

Then $x + 3 =$ the number of feet in the length of the rectangle,

and $x - 2 =$ the number of feet in the width of the rectangle.

$\therefore (x + 3)(x - 2) =$ the area of the rectangle in square feet,

and $x^2 =$ the area of the square in square feet.

$\therefore (x + 3)(x - 2) = x^2.$

Solving, $x = 6$, the number of feet in the side of the square.

$\therefore x + 3 = 9$, the number of feet in the length of the rectangle,

and $x - 2 = 4$, the number of feet in the width of the rectangle.

6. A rectangle is l feet longer and n feet narrower than a square of the same area. Find the side of the square; the length of the rectangle; the width of the rectangle.

7. The length of a floor exceeds the width by 2 ft. If each dimension were 4 ft. more, the area would be 136 sq. ft. more. Find the dimensions.

8. The length of a floor exceeds the width by f feet. If each dimension were n feet more, the area would be s square feet more. Find the dimensions. Evaluate for $f = 2$, $n = 4$, $s = 136$.

9. The length of a rectangular field is 10 rd. more than the width. If the width were 6 rd. more and the length were 5 rd. more, the area would be 200 sq. rd. more. Find the dimensions of the field.

10. Two stations are 300 mi. apart. From the first of these stations a train traveling 30 mi. an hour leaves at the same time that a train traveling 40 mi. an hour leaves the second station. How far apart will the trains be at the end of 3 hr. if they travel toward each other?

Let $x =$ the number of miles they are apart after 3 hr.

Since they approach at the rate of $(30 + 40)$ miles an hour, in 3 hr. they are $3 \cdot (30 + 40)$ miles nearer each other.

$$\therefore 3 \cdot (30 + 40) + x = 300.$$

Solving, $x = 90.$

11. In Ex. 10, if the trains travel away from each other, how far apart will they be in 3 hr.?

12. Two stations are d miles apart. From the first of these stations a train traveling r_1 miles an hour leaves at the same time that a train traveling r_2 miles an hour leaves the second station. How far apart will the trains be at the end of t hours if they travel toward each other?

13. In Ex. 12, if the trains travel away from each other, how far apart will they be in t hours?

14. What is the result in Ex. 12 if $r_1 = 30$, $r_2 = 40$, $t = 3$, and $d = 300$? also, if $d = 210$?

15. What is the result in Ex. 12 if $r_1 = 30$, $r_2 = 40$, $t = 3$, and $d = 240$? also, if $d = 180$?

16. What is the result in Ex. 13 if $r_1 = 30$, $r_2 = 40$, $t = 3$, and $d = 300$? also, if $d = 210$?

17. What is the result in Ex. 13 if $r_1 = 30$, $r_2 = 40$, $t = 3$, and $d = 240$? also, if $d = 180$?

18. What are the results in Exs. 12 and 13 if $r_1 = 30$, $r_2 = 30$, $t = 2$, and $d = 60$?

19. A man walking 3 mi. an hour has 5 mi. the start of one walking 4 mi. an hour. How long will it take the second to overtake the first? If the first walks r_1 miles an hour and the second r_2 miles an hour $(r_1 < r_2)$, and the first has m miles the start, how long will it take the second to overtake the first?

20. A can do a piece of work in 4 da. and B can do it in 5 da. How many days will it take both together to do the work?

Let x = the number of days it will take both together.

Then $\frac{1}{x}$ = the part that both together can do in 1 da.,

$\frac{1}{4}$ = the part that A alone can do in 1 da.,

$\frac{1}{5}$ = the part that B alone can do in 1 da.,

and $\frac{1}{4} + \frac{1}{5}$ = the part that both together can do in 1 da.

$$\therefore \frac{1}{4} + \frac{1}{5} = \frac{1}{x}.$$

Solving, $x = 2\frac{2}{9}$.

21. A can do a piece of work in a days and B can do it in b days. How many days will it take both together to do the work? Evaluate the result for $a = 4$, $b = 5$.

22. A can do a piece of work in 4 da., B in 5 da., and C in 6 da. How many days will it take them to do it working together?

23. A can do a piece of work in a days, B in b days, and C in c days. How many days will it take them to do it working together? Evaluate the result for $a = 4$, $b = 5$, $c = 6$.

24. Evaluate the first result in Ex. 23 for $a = 2\frac{1}{2}$, $b = 3\frac{1}{3}$, $c = 3\frac{3}{4}$; also for $a = 6$, $b = 5$, $c = 4$.

25. A and B together can build a wall in 12 da., A and C in 15 da., B and C in 20 da. In how many days can they build the wall if they all work together?

By working 2 da. each they build $\frac{1}{12} + \frac{1}{15} + \frac{1}{20}$ of it. Why?
Hence in 1 da. they build $\frac{1}{2}(\frac{1}{12} + \frac{1}{15} + \frac{1}{20})$ of it.

26. A and B together can build a wall in 10 da., A and C in 12 da., and B and C in 15 da. In how many days can they build the wall if all work together?

27. A and B together can build a wall in d days, A and C in e days, B and C in f days. In how many days can they build the wall if they all work together? Evaluate the result for $d = 12$, $e = 15$, $f = 20$; for $d = 10$, $e = 12$, and $f = 15$.

28. A tank can be filled by two pipes in 2 hr. and 3 hr., respectively. In how many hours can it be filled if both pipes are open together?

Let $x =$ the number of hours to fill the tank if both pipes are open together.

Then $\frac{1}{x} =$ the part filled in 1 hr. if both pipes are open together.

But $\frac{1}{2} + \frac{1}{3} =$ the part filled in 1 hr. if both pipes are open together.

$$\therefore \frac{1}{2} + \frac{1}{3} = \frac{1}{x}.$$

Solving, $x = 1\frac{1}{5}$.

$1\frac{1}{5}$ hr. = 1 hr. 12 min.

29. A tank can be filled by two pipes in 2 hr. and 3 hr., respectively, and emptied by a waste pipe in 4 hr. In how many hours can it be filled if all three pipes are open together?

Let $x =$ the number of hours if all three pipes are open.

Then $\frac{1}{x} =$ the part filled in 1 hr. if all three pipes are open.

But $\frac{1}{2} + \frac{1}{3} - \frac{1}{4} =$ the part that all can fill in 1 hr.

30. A tank can be filled by three pipes in a, b, and c hours, respectively. In how many hours can it be filled if all three pipes are open together?

31. Evaluate the result in Ex. 30 for $a = 8$, $b = 12$, $c = 16$; for $a = 1\frac{2}{3}$, $b = 3\frac{1}{3}$, $c = 5$; for $a = 2\frac{1}{3}$, $b = 3\frac{1}{2}$, $c = 4\frac{2}{3}$.

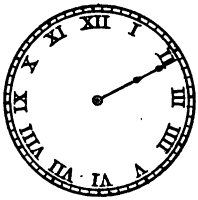
32. A tank can be filled by two pipes in a hours and b hours, respectively, and emptied by a waste pipe in c hours. In how many hours can it be filled if all three pipes are open together? Evaluate the result for $a = 2$, $b = 3$, $c = 4$.

33. Evaluate the first result in Ex. 32 for $a = 4$, $b = 5$, $c = 6$; for $a = 10$, $b = 12$, $c = 15$.

34. A cistern has three pipes. The first will fill the cistern in 6 hr., the second in 10 hr., and all three together in 3 hr. How long will it take the third pipe alone to fill it?

35. Solve Ex. 34, using a, b, and c instead of 6, 10, and 3.

36. Find the time between 2 and 3 o'clock when the hands of a clock are together.



Let x = the number of minute spaces passed over by the hour hand.

Then $12x$ = the number passed over by the minute hand.

Then $12x = x + 10$, since the hour hand is 10 spaces ahead at 2 o'clock.

$\therefore x = \frac{10}{11}$, and $12x = \frac{120}{11} = 10\frac{10}{11}$.

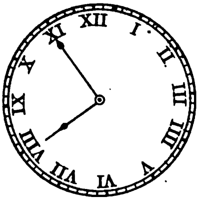
Therefore they will be together at $10\frac{10}{11}$ min. after 2.

Find the time when the hands of a clock are together between:

37. 3 and 4. **39.** 5 and 6. **41.** 9 and 10.

38. 4 and 5. **40.** 6 and 7. **42.** 11 and 12.

43. Find the time between 7 and 8 o'clock when the hands of the clock are at right angles to each other.



Let x = the number of minute spaces passed over by the hour hand.

Then $12x$ = the number passed over by the minute hand.

But the minute hand must gain 35 spaces to catch up with the hour hand after 7 o'clock, or $35 + 15$ spaces to be 15 spaces (a right angle) ahead, or $35 - 15$ spaces to be 15 spaces behind it.

Then $12x = x + 50$, when the minute hand is 15 spaces ahead,

or $12x = x + 20$, when it is 15 spaces behind.

$\therefore x = \frac{50}{11}$ or $\frac{20}{11}$,

and $12x = \frac{600}{11} = 54\frac{6}{11}$, or $\frac{240}{11} = 21\frac{9}{11}$.

But $54\frac{6}{11}$ min. = 54 min. $32\frac{8}{11}$ sec., and $21\frac{9}{11}$ min. = 21 min. $49\frac{1}{11}$ sec.

Therefore the time is 7 hr. 54 min. $32\frac{8}{11}$ sec.

or 7 hr. 21 min. $49\frac{1}{11}$ sec.

Find the time when they are at right angles between:

44. 6 and 7. **45.** 2 and 3. **46.** 11 and 12.

Find the time when they are opposite between:

47. 4 and 5. **48.** 10 and 11. **49.** 9 and 10.

50. Find two consecutive numbers, x and $x + 1$, such that $\frac{1}{2}$ of the smaller exceeds $\frac{1}{3}$ of the larger by 5.

51. Find three consecutive numbers such that their sum is 99; such that their sum is 159.

52. Find three consecutive numbers such that the sum of the largest and smallest is 20 more than the other number.

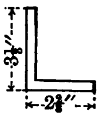
53. Find four consecutive numbers such that their sum is 126; such that their sum is 210.

54. Find four consecutive numbers such that one sixth of the smallest is two more than one eleventh of the largest.

55. The difference of two numbers is 30, and if the greater is divided by the less the quotient is 11. Find the numbers.

56. The difference of two numbers is 12, and if the greater is divided by the less the quotient is 2 and the remainder 5. Find the numbers.

57. A man has 6 hr. at his disposal. How far may he ride in a trolley car at 9 mi. an hour so as to return in time, walking back at the rate of 3 mi. an hour?



58. The number of square inches of area of a cross section of an L beam is 3 minus the square of the thickness. Find the thickness.

59. A train that travels 36 mi. an hour is 45 min. ahead of a second train that travels 42 mi. an hour. How long will it take the faster train to overtake the slower?

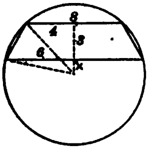
60. If to x sevenths we add $\frac{5}{8}$, the result is $1\frac{43}{56}$. Find the value of x.

61. If to seven $x$ths we add $\frac{5}{8}$, the result is $1\frac{1}{2}$. Find the value of x.

62. If we divide 8 by $x + 1$ the result is the same as that of dividing 4 by $x - 1$. Find the value of x.

63. If to twice a certain number we add the number itself, the result is the number increased by 6. What is the number?

64. The bases of a trapezoid inscribed in a circle are 12 in. and 8 in., respectively, and the altitude is 3 in. Find the distance from the center to the lower base, and find the radius.



Show that $x^2 + 6^2 = (x + 3)^2 + 4^2$.

65. A caterer prepares a dinner for 50 people at a cost of $50 to himself. How much must he charge per plate to be sure of making $17.80 if only 45 people come, counting the unused food as worth 20¢ per plate?

66. A boy walked from A to B at the rate of $3\frac{1}{2}$ mi. per hour. He returned immediately on horseback at the rate of $4\frac{2}{3}$ mi. an hour. Upon his return he found that he had been gone $3\frac{1}{2}$ hr. How far is it from A to B?

67. A man purchases ice at 25¢ per 100 lb. At what rate must he sell it after it has lost 10% of its weight by melting, so that he may gain 17%?

68. A man purchases ice at c cents per 100 lb. At what rate must he sell it after it has lost $r\%$ of its weight by melting, so that he may gain $g\%$?

69. Calculate the side of an equilateral triangle of which the altitude is h. Write a formula for the area of such a triangle.

70. Calculate the sides of an isosceles triangle of which the perimeter is $2p$ and the altitude is h. Evaluate the result for $p = 4$, $h = 2$.

71. A man pays $60 a month for his city apartment and $20 a month for service, the heating being furnished free. If he moves to the suburbs, his car fare will be $8 a month, his coal bill will average $7 a month, and his service will cost one third as much as his rent. How much rent can he afford to pay in the suburbs so that the total expense per month will be the same as in the city?

72. Alcohol is sold 95% pure. How much water must be added to 1 qt. of such alcohol so that the mixture will be 75% pure?

Let x = the number of quarts to be added.

Then $0.75(1 + x)$ = the number of quarts of alcohol in the mixture.

But 0.95 = the original number of quarts of alcohol, and none has been added.

$$\therefore\ 0.75(1 + x) = 0.95.$$

Dividing by 0.75,

$$1 + x = \tfrac{19}{15}.$$

Subtracting 1, $x = \tfrac{4}{15}$.

Therefore $\tfrac{4}{15}$ qt. must be added.

73. How much water must be added to a 10% solution of a certain medicine to reduce it to a 2% solution?

74. How many ounces of pure silver must be melted with 75 oz. of silver 750 fine (750 parts of pure silver in 1000 parts of metal) to make a piece of silver 900 fine?

75. How many ounces of gold must be melted with 20 oz. of gold 16 carats fine ($\tfrac{16}{24}$ pure) to make a piece of gold 20 carats fine?

76. How many quarts of milk containing 5% butter fat must be added to 1 qt. of cream containing 35% butter fat so that the mixture will contain 25% butter fat?

77. In a certain alloy of metal weighing 25 lb., 18.75% is pure silver. How many pounds of copper must be melted with it so that the new alloy will contain $15\tfrac{5}{8}$% pure silver?

78. How many pounds of pure water must be added to 25 lb. of sea water containing 16% of salt so that the mixture will contain 2% of salt?

79. What per cent of water must be evaporated from a 6% solution of salt so that the remainder will be a 10% solution?

80. What per cent of water must be evaporated from an r% solution of salt so that the remainder will be an r'% solution?

CHAPTER XIII

RATIO, PROPORTION, AND VARIATION

156. Ratio. The relation of one number to another number of the same kind, as indicated by division, is called the *ratio* of the first to the second.

The ratio of $a + b$ to $c + d$ is indicated thus: $\dfrac{a+b}{c+d}$ or $a + b : c + d$.

Hence all ratios may be looked upon as fractions.

157. Terms of a Ratio. The two numbers involved in a ratio are called the *terms* of the ratio. The dividend is called the *antecedent* and the divisor is called the *consequent.*

Thus we have

$$\frac{a}{b} = \frac{\text{antecedent}}{\text{consequent}} = \frac{\text{numerator}}{\text{denominator}} = \frac{\text{dividend}}{\text{divisor}}.$$

If $\dfrac{a}{b} = \dfrac{2}{3}$, we say that a is to b as 2 is to 3.

158. Greater and Less Inequality. If the absolute value of a ratio is greater than 1, the ratio is called a *ratio of greater inequality.* If the absolute value of a ratio is less than 1, the ratio is called a *ratio of less inequality.*

Thus $\frac{3}{2}$ is a ratio of greater inequality, and $\frac{2}{3}$ is a ratio of less inequality. The symbol $>$ is read "is greater than," and the symbol $<$ is read "is less than."

159. Laws of Ratio. The following are the important laws of a ratio whose terms are positive.

1. *Both terms of a ratio may be multiplied, or both may be divided, by the same number without changing the value of the ratio.*

A ratio being a fraction, the laws of fractions are true for ratios.

2. *A ratio of greater inequality is diminished, and a ratio of less inequality is increased, by adding the same positive number to both terms.*

Let a be the antecedent and b the consequent.

Then $\quad \dfrac{a+n}{b+n} >$ or $< \dfrac{a}{b},$

according as $\quad ab + bn >$ or $< ab + an,$

or as $\quad bn >$ or $< an,$

or as $\quad b >$ or $< a.$

That is, if $\quad a > b,$ then $\dfrac{a+n}{b+n} < \dfrac{a}{b},$

and if $\quad a < b,$ then $\dfrac{a+n}{b+n} > \dfrac{a}{b}.$

3. *A ratio of greater inequality is increased, and a ratio of less inequality is decreased, by subtracting the same positive number from both terms.*

We must first notice that

$$-4 > -5, \text{ but } 4 < 5,$$

and that, in general, if $\quad -b > -a,$ then $b < a.$

That is, changing the sign changes the order of inequality.

We see that $\quad \dfrac{a-n}{b-n} >$ or $< \dfrac{a}{b},$

according as $\quad ab - bn >$ or $< ab - an,$

or as $\quad -bn >$ or $< -an,$

or as $\quad b <$ or $> a.$

4. *In a series of equal ratios the sum of all the antecedents is to the sum of all the consequents as any antecedent is to its consequent.*

Consider the three ratios $\dfrac{a}{b}, \dfrac{c}{d}, \dfrac{e}{f}$, and let each equal r.

Since $\quad \dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = r, \therefore a = br, c = dr,$ and $e = fr.$

Adding, $\quad a + c + e = (b + d + f)r.$

Dividing, $\quad \dfrac{a+c+e}{b+d+f} = r = \dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}.$

Exercise 115. Ratios

Examples 1 to 4, oral — Examples 5 to 17, written

1. Express in simplest fractional form the ratio of 10 to 20; of 32 to 40; of 45 to 54; of 49 to 56; of 26 to 39.

2. Express in simplest form, as improper fractions, the ratio of 15 to 10; of 20 to 12; of 35 to 21; of 81 to 63.

3. How is the ratio of 9 to 5 changed by multiplying both terms by 2? by adding 2 to both terms? by subtracting 2 from both terms?

4. How is the ratio of 6 to 9 changed by multiplying both terms by 3? by dividing both terms by 3? by adding 3 to both terms? by subtracting 3 from both terms?

Simplify the following ratios:

5. $a^2 - b^2 : a^3 - b^3.$

6. $\dfrac{1}{x+y} : \dfrac{1}{x^2 + 2xy + y^2}.$

7. $a^3 - 1 : a^2 + a + 1.$

8. $1 + \dfrac{a}{b} : 1 - \dfrac{a}{b}$

9. Separate 50 into two parts having the ratio 2 : 3.

Let $x =$ the smaller part.

Then $50 - x =$ the larger part.

Then $$\frac{x}{50 - x} = \frac{2}{3}.$$

Therefore $x = 20$, and $50 - x = 30$.

Separate into two parts having the ratios specified:

10. 24, 1 : 2.

11. 35, 2 : 3.

12. 91, 2 : 5.

13. 104, 3 : 5.

14. 272, 7 : 9.

15. 306, 7 : 11.

16. A wheel 30 in. in diameter, making 300 revolutions per minute, is belted to another wheel 15 in. in diameter. Find the speed of the smaller wheel.

17. Two cogwheels are geared together. The distance between their centers is 20 in. What are their diameters if their speeds have a ratio of 4 to 5?

160. Proportion. An expressed equality of two ratios is called a *proportion.*

Thus if $\frac{a}{b}=\frac{c}{d}$, then a, b, c, d are said to be in proportion or to form a proportion.

This proportion is often read, "a is to b as c is to d," and may be written thus: $a:b=c:d$. In older works it is often written $a:b::c:d$.

161. Terms of a Proportion. In a proportion the first and fourth terms are called the *extremes,* and the second and third terms the *means.*

In the proportion $a:b=c:d$ the extremes are a and d, and the means are b and c.

In the same proportion d is called the *fourth proportional* to a, b, and c.

162. Mean Proportional. If the means of a proportion are the same, this term is called the *mean proportional* between the other two.

Thus m is the mean proportional between x and y if $x:m=m:y$. In this proportion y is called the *third proportional* to x and m.

163. Laws of Proportion. The following are the important laws of proportion:

1. *If four numbers are in proportion, the product of the extremes is equal to the product of the means.*

Let the proportion be $\frac{a}{b}=\frac{c}{d}$.

Multiplying by bd, $ad=bc$.

2. *If the product of two numbers equals the product of two other numbers, either two may be made the means of a proportion, and the other two the extremes.*

Let $ad=bc$.

Dividing by bd, $\frac{ad}{bd}=\frac{bc}{bd}$,

or $\frac{a}{b}=\frac{c}{d}$.

3. *If $a : b = c : d$, then $b : a = d : c$.*

In this case the first proportion is said to be taken by *inversion.*

From Law 1, $bc = ad.$

Dividing by ac, $$\frac{bc}{ac} = \frac{ad}{ac},$$

or $$\frac{b}{a} = \frac{d}{c}.$$

4. *If $a : b = c : d$, then $a : c = b : d$.*

In this case the first proportion is said to be taken by *alternation.*

From Law 1, $ad = bc.$

Dividing by cd, $$\frac{ad}{cd} = \frac{bc}{cd},$$

or $$\frac{a}{c} = \frac{b}{d}.$$

5. *If $a : b = c : d$, then $a + b : b = c + d : d$.*

In this case the first proportion is said to be taken by *composition.*

Given $$\frac{a}{b} = \frac{c}{d}.$$

Adding 1, $$\frac{a}{b} + 1 = \frac{c}{d} + 1,$$

or $$\frac{a+b}{b} = \frac{c+d}{d}.$$

6. *If $a : b = c : d$, then $a - b : b = c - d : d$.*

In this case the first proportion is said to be taken by *division.*

Given $$\frac{a}{b} = \frac{c}{d}.$$

Subtracting 1, $$\frac{a}{b} - 1 = \frac{c}{d} - 1,$$

or $$\frac{a-b}{b} = \frac{c-d}{d}.$$

7. *The product of the means divided by either extreme equals the other extreme. The product of the extremes divided by either mean equals the other mean.*

The proof is left for the student.

Exercise 116. Proportion

Examples 1 to 6, oral — Examples 7 to 38, written

1. Solve the equation $\frac{1}{2}x = \frac{2}{3}$.

2. In the proportion $\frac{x}{2} = \frac{2}{3}$, find the value of x.

3. In the proportion $x : 2 = 2 : 3$, find the value of x.

Find the value of x in the following proportions:

4. $\frac{x}{4} = \frac{3}{8}$. **7.** $\frac{2}{3} = \frac{x}{9}$. **10.** $\frac{7}{x} = \frac{42}{48}$.

5. $\frac{x}{5} = \frac{7}{15}$. **8.** $\frac{3}{5} = \frac{12}{x}$. **11.** $\frac{36}{48} = \frac{3}{x}$.

6. $\frac{x}{7} = \frac{11}{21}$. **9.** $\frac{5}{x} = \frac{25}{35}$. **12.** $\frac{65}{39} = \frac{5}{x}$.

13. If the means are 3 and 12, and one extreme is 4, what is the other extreme?

14. If $x - 5 : 5 = 28 : 10$, by what law do we know that $x : 5 = 38 : 10$? What is the value of x?

Find the value of x in the following proportions:

15. $x : 2.4 = 1.21 : 1.1$. **17.** $2.6 : x = 1.1 : 7.7$.

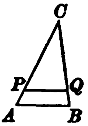
16. $x - 12 : 12 = 5 : 1$. **18.** $17 - x : x = 16 : 18$.

19. If each mean of a proportion is 8, and one extreme is 32, what is the other extreme?

20. In a lever PW, if sufficient power (p) is applied at P, a weight (w) at W can be lifted. If F is the fulcrum, it is known that $p : w$ equals the ratio $FW : FP$. If $FW = 10$ in., $FP = 30$ in., and $w = 240$ lb., what does p equal?

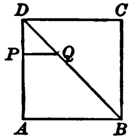
21. If two boys weigh respectively 100 lb. and 120 lb., where must the fulcrum be placed under a 10-foot board so that the boys sitting at the ends will just balance? (Use the law given in Ex. 20.)

22. It is proved in geometry that a line parallel to the base of a triangle divides the other two sides proportionally. In this figure, if $AP = 1\frac{1}{2}$, $PC = 6$, and $QC = 5$, find the length of BQ.



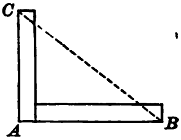
23. If a line parallel to the base of a triangle divides one side into the parts $7\frac{1}{2}$ and $4\frac{1}{4}$, and the other side into the corresponding parts 6 and x, what is the value of x?

24. If two sides of a triangle are 7 and 9, and a line parallel to the base cuts the former into the parts 3 and 4, into what lengths will it cut the latter?

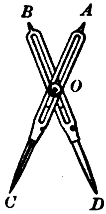


25. In the square here shown, PQ is ∥ to AB. If a side of the square is 10 in., $DB = 14.14$ in. If $DP = 3$ in., what is the length of DQ?

26. Two pieces of timber 1 ft. wide are fitted together at right angles as here shown. AB is 8 ft. long, AC 6 ft. long, and the distance BC along the dotted line is 10 ft. A carpenter finds it necessary to saw along the dotted line. Find the length of the slanting cut across the upright piece; across the horizontal piece. $x : 10 = 1 : 6$ is one proportion.



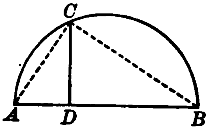
27. It is proved in geometry that two triangles of the same shape have their corresponding sides proportional. Such triangles are said to be *similar*. In Ex. 22 the triangles ABC and PQC are similar. If $AB = 8$, $AC = 7\frac{1}{2}$, and $PC = 6$, what is the length of PQ?



28. The figure at the right represents a pair of proportional compasses used by draftsmen. By adjusting the screw at O, the lengths OA and OC, and the corresponding lengths OB and OD, may be varied proportionally. The triangle formed by O, A, and B is always similar to the triangle formed by O, C, and D. If $OA = 3$ in. and $OC = 5$ in., then AB is what part of CD?

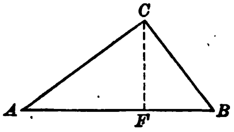
29. In the following combination of levers a power of 100 lb. is applied at P. What weight can be raised at W?

30. It is proved in geometry that if AB is the diameter of the semicircle here shown, CD is a mean proportional between AD and DB. If $AD = 3$ and $CD = 4$, what is the length of DB?



31. In the same figure, if $DB = 4AD$ and $CD = 12$, what are the lengths of AD and DB?

32. If a perpendicular is let fall from the vertex of the right angle upon the hypotenuse of a right triangle, it divides the right triangle into two triangles similar to the original triangle and to each other. In the given figure, if $AF = 5$ and $CF = 3.5$, what is the length of FB?



33. In the same figure, if $AB = 7.45$ and $AC = 6.1$, what is the length of AF?

34. If a boy $4\frac{3}{4}$ ft. tall casts a shadow $4\frac{1}{2}$ ft. long at the same time that the school building casts a shadow $67\frac{1}{2}$ ft. long, how high is the school building?

35. If a boy, lying down with his eye to the ground, sights over the top of a 10-foot pole held vertically $6\frac{1}{4}$ ft. from his eye, and can just see the top of a tree $37\frac{1}{2}$ ft. from his eye, how tall is the tree?

36. Show that if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$. This is called the taking of the original proportion by *composition and division.*

37. Solve the equation $\frac{x+2}{x-2} = \frac{10}{7}$ by the principle of Ex. 36.

38. Solve the equation $\frac{x-7}{x+7} = \frac{2}{3}$ by the principle of Ex. 36.

164. Variation. If two variable quantities, x and y, have a constant ratio k, either quantity is said to *vary* as the other.

If $\frac{x}{y} = k$, then $x = ky$, and $y = \frac{1}{k} \cdot x$.

The expression "x varies as y" is sometimes written $x \propto y$, which means that $x = ky$, k being the constant ratio.

For example, in a circle $c = 2\pi r$, so that $c \propto r$, c always being 2π times r.

Similarly, for the area of a circle, $a = \pi r^2$, in which case $a \propto r^2$.

The subject of variation required in certain courses of study in connection with ratio may be omitted without affecting the subsequent work.

If $x \propto y$, and $x = 12$ when $y = 4$, find x when $y = 6$.

Since $x \propto y$, therefore $x = ky$, where k is some constant.

Hence $12 = k \cdot 4$,

and therefore $3 = k$.

Hence, when $y = 6$,

$$x = k \cdot 6 = 3 \cdot 6 = 18.$$

Exercise 117. Variation

Examples 1 to 3, oral — Examples 4 to 10, written

1. If $x = ky$, find k when $x = 21$ and $y = 10.5$.
2. If $x = ky$, find y when $x = 27$ and $k = 4.5$.
3. If $a = \pi r^2$, find a when $\pi = 3.1416$ and $r = 100$.
4. If $x \propto y$, and $x = 19$ when $y = 7$, find x when $y = 10.5$.
5. If $x \propto y$, and $x = 17$ when $y = 9$, find y when $x = 25.5$.
6. If $x \propto y$, and the ratio of x to y is $\frac{3}{4}$, find x when $y = 28$.
7. In Ex. 6 find y when $x = 36$; when $x = 4.83$.
8. If $x = ky$, and $y = mz$, find x in terms of z.
9. If $x \propto y$, and $y \propto z$, show that $x \propto z$.
10. If the circumference of a circle varies as the radius, and the radius varies as the diameter, show that the circumference varies as the diameter.

165. Joint Variation. The area of a triangle varies as the base when the altitude is constant, and as the altitude when the base is constant. If both base and altitude vary, the area varies as their product.

In other words, if the base remains unchanged, the area will be doubled by doubling the altitude. Likewise, if the altitude remains unchanged, the area will be doubled by doubling the base. But if we double both base and altitude, the area will be multiplied by 4.

That is, $a \propto bh$, or $a = kbh$. In fact, as we know (§ 6), $a = \frac{1}{2}bh$, so that in this case $k = \frac{1}{2}$. In general, if $x \propto y$ when z is unchanged, and $x \propto z$ when y is unchanged, then $x \propto yz$ when both change.

166. Inverse Variation. If two variables, x and y, are so related that the ratio of x to $\frac{1}{y}$ is some constant number, k, either quantity is said to *vary inversely* as the other.

If $x : \frac{1}{y} = k$, then $xy = k$, or $x = \frac{k}{y}$, and $y = \frac{k}{x}$.

As y increases, x decreases at the same rate. If y is made twice as large, x becomes half as large.

For example, if a certain number of men, m, can do a piece of work in t days, then twice as many men can do the work in $\frac{1}{2}t$ days, and $\frac{1}{4}$ as many men can do it in $4t$ days. That is, $m \propto \frac{1}{t}$.

1. If $x \propto \frac{1}{y}$, and $x = 9$ when $y = 7$, find x when $y = 2$.

Since $x \propto \frac{1}{y}$, we have $xy = k$.

Hence $9 \cdot 7 = 63 = k$.

Hence $x \cdot 2 = 63$, when $y = 2$,

and $x = 31.5$.

2. If x varies directly as y and inversely as z, and if $x = 12$ when $y = 6$ and $z = 2$, find x when $y = 8$ and $z = 1$.

Since $x \propto y$, and $x \propto \frac{1}{z}$, then $x = ky \cdot \frac{1}{z}$ (§ 165).

Hence $12 = k \cdot 6 \cdot \frac{1}{2}$,

and $k = 4$.

Then $x = 4 \cdot 8 \cdot 1 = 32$,

when $y = 8$ and $z = 1$.

Exercise 118. Problems in Variation

All Examples written

1. If x varies inversely as y, and $x = 12$ when $y = 2$, find x when $y = 7$.

2. If $x \propto y$, and $x \propto \frac{1}{z}$, and if $x = 6$ when $y = 8$ and $z = 12$, find x when $z = 28$ and $y = 4$.

3. If $x \propto y$, and $x \propto \frac{1}{z}$, and if $x = 15$ when $y = 3$ and $z = 5$, find y when $x = 35$ and $z = 12.5$.

4. If $y \propto \frac{1}{x}$, and $y \propto \frac{1}{z}$, and if $x = 2$ when $y = 4$ and $z = 3$, find z when $x = 4$ and $y = 1$.

5. If $xy = a$, and $yz = b$, and if $x = 4$ when $y = 8$ and $z = 6$, find z when $x = 8$ and $y = 2$.

6. The weight of a sphere of steel varies as the volume, and the volume varies as the cube of the diameter. If a sphere 1 in. in diameter weighs 0.15 lb., find the weight of a sphere 4 in. in diameter.

Representing the weight, volume, and diameter by w, v, and d, respectively, we have $w \propto v$, and $v \propto d^3$.

Hence $w = mv$, and $v = nd^3$,

so that $w = mnd^3$;

or, we may say, $w = kd^3$.

Therefore $0.15 = k \cdot 1^3 = k$.

Hence, when $d = 4$, $w = 0.15 \cdot 4^3 = 9.6$.

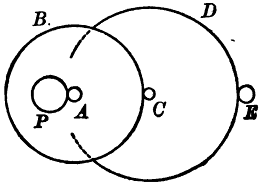
Hence a sphere of steel 4 in. in diameter weighs 9.6 lb.

7. The diagonal of a cube varies as the edge. When the edge is 6 in. the diagonal is 10.4 in. Find the diagonal when the edge is 15 in.

8. The capacity of a water main varies as its length and also as the square of its diameter. If a pipe 1 ft. in diameter and 16 ft. long has a capacity of 12.576 cu. ft., what is the capacity of a water main 10 in. in diameter and 480 ft. long?

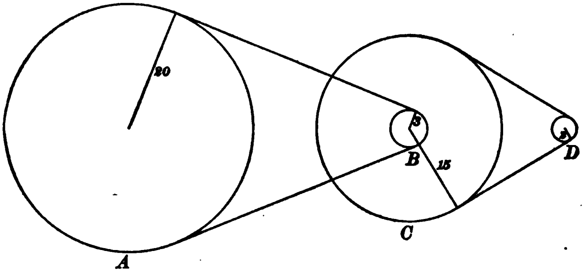
9. The value of a diamond varies as the square of the weight. If a diamond worth $640 is cut into two pieces whose weights are as 1 to 3, what is the value of each piece?

10. The figure at the right represents several wheels geared together. P has 50 teeth, A 20 teeth, B 200 teeth, C 15 teeth, D 250 teeth, and E 25 teeth. If P makes 25 revolutions per minute, how many revolutions per minute will E make?



11. A cube of clay 14 in. on an edge is molded into a right prism whose base is a square 7 in. on a side. Find the height of the prism.

12. The intensity of light varies inversely as the square of the distance. How far from a lamp is a point that receives half as much light as another point 16 ft. from the lamp?



13. Several wheels are belted together as here shown. The circumferences of circles varying as the radii, and the radii being as indicated, how many revolutions per minute will D make when A is making 2000 revolutions per minute?

14. The volume of a right circular cylinder varies as the product of its height and the square of its radius. A cylindric can is 5 in. high and 1.5 in. in radius, and another is 6 in. high and has a radius of 2 in. Find the radius of a third can 7 in. high that will hold as much as the other two together.

CHAPTER XIV

SIMULTANEOUS SIMPLE EQUATIONS

167. Indeterminate Equations. If we have one simple equation containing one unknown quantity, we can solve by the methods already studied. If it contains two unknown quantities, we can find one quantity in terms of the other.

For example, if $2x - 3y = 8$, we can see that $x = \frac{8 + 3y}{2}$, and that $y = \frac{2x - 8}{3}$; but this does not tell us the value of either.

Indeed, if $y = 1$ we see that $x = 5\frac{1}{2}$; if $y = 2$, $x = 7$; if $y = 4$, $x = 10$, and so on, there being an indefinite number of values of x and y that satisfy this equation.

An equation that has an indefinite number of roots is called an *indeterminate equation.*

An equation that is not indeterminate is said to be *determinate.*

168. Two Simple Equations. If we have two simple equations containing two unknown quantities, we can usually derive from them a single equation containing only one unknown quantity.

For example, suppose $x + y = 17,$

and $x - y = 9.$

Adding member for member, $2x = 26;$

whence $x = 13.$

And because $x + y = 17,$

we have $13 + y = 17.$

Whence $y = 4.$

169. Simultaneous Equations. Two or more equations that have the same values for the unknown quantities are called *simultaneous equations.*

170. Elimination. The process by which we cause an unknown quantity to disappear in deriving an equation from a system of equations is called *elimination.*

Thus in § 168 we derived the equation $2x = 26$ from the system of equations $x + y = 17$ and $x - y = 9$, thereby eliminating y.

The most common methods of elimination are (1) by addition (or subtraction as a special case), and (2) by substitution. These will now be considered.

171. Elimination by Addition or Subtraction. This process is best understood by studying two examples.

1. Solve the system of equations

$$2x + 3y = 27 \tag{1}$$

$$5x - 2y = 1 \tag{2}$$

Multiplying (1) by 2, $4x + 6y = 54$

Multiplying (2) by 3, $\underline{15x - 6y = 3}$

Adding, $19x = 57$

Dividing by 19, $x = 3.$

Substituting 3 for x in (1), $6 + 3y = 27.$

Subtracting 6, $3y = 21.$

Dividing by 3, $y = 7.$

Check. Substituting 3 for x, and 7 for y, in (1) and (2), we have

$$6 + 21 = 27,$$

$$15 - 14 = 1.$$

Because y was eliminated by adding two equations, member for member, we say that we have eliminated y by *addition.*

2. Solve the system of equations

$$3x + 2y = 23 \tag{1}$$

$$2x + 3y = 27 \tag{2}$$

Multiplying (1) by 3, $9x + 6y = 69$ (3)

Multiplying (2) by 2, $\underline{4x + 6y = 54}$ (4)

Subtracting (4) from (3), $5x = 15$

Dividing by 5, $x = 3.$

Substituting in (1) or (2), $y = 7.$

In this solution y is eliminated by *subtraction.*

172. Directions for Elimination. From the preceding solutions we see that in the elimination of an unknown quantity by addition or subtraction we proceed as follows:

Multiply both members of the equations by such numbers as will make the coefficients of one of the unknown quantities numerically equal.

If these coefficients have opposite signs, add the equations member for member; if they have the same signs, subtract.

Solve the resulting equation.

Substitute the result thus found in the simpler of the two given equations and solve for the other unknown quantity.

Check the results by substituting in both of the given equations.

We commonly say "Multiply the equation," meaning thereby "Multiply both members of the equation." Similarly, we speak of adding equations and subtracting equations, meaning that we add or subtract member for member.

Exercise 119. Elimination by Addition or Subtraction

Examples 1 to 8, oral — Examples 9 to 38, written

1. Solve for x,

$$x + y = 9$$
$$x - y = 3$$

2. Solve for y,

$$x + 2y = 7$$
$$x + y = 4$$

3. Solve for x,

$$3x + 2y = 5$$
$$5x - 2y = 3$$

4. Solve for m,

$$7m + 3n = 23$$
$$5m + 3n = 19$$

5. Solve for u,

$$2u + v = 14$$
$$3u - v = 11$$

6. Solve for w,

$$2w + 3z = 20$$
$$w - 3z = 1$$

7. Solve for P,

$$3P + Q = 17$$
$$2P - Q = 8$$

8. Solve for F,

$$2F + 7G = 21$$
$$3F - 7G = 14$$

Solve the following by addition or subtraction:

9. $2x + 3y = 13$
$5x - 4y = 16$

10. $2x + 5y = 51$
$5x + 2y = 54$

11. $6x + 5y = 33$
$7x - 2y = 15$

12. $9x + 7y = 16$
$3x - 2y = 1$

13. $9x + 2y = 24$
$7x + 3y = 23$

14. $4x + y = 23$
$5x - y = 13$

15. $7x + 6y = 20$
$5x - 2y = 8$

16. $4x + 9y = 3$
$3x + 7y = 2$

17. $3x + 8y = 38$
$7x - 2y = 6$

18. $5x - 3y = 20$
$3x - 4y = 1$

19. $4x + 3y = 18$
$3x - 2y = 5$

20. $5x + 2y = 32$
$3x + y = 18$

21. $2x + 7y = 34$
$7x + 2y = 74$

22. $4x - 7y = 20$
$7x - 4y = 68$

23. $21x + 11y = 288$
$30x - 16y = 126$

24. $0.2x + 0.3y = 12$
$0.5x + 0.4y = 23$

25. $0.7x + 0.5y = 53$
$0.9x - 0.4y = 16$

26. $1.5x + 1.25y = 19$
$1.2x + 0.75y = 13.2$

27. $u + 4v = 31$
$4u + v = 19$

28. $8v + 5w = 59$
$7v + 2w = 35$

29. $5m - 2n = 23$
$13m - 3n = 51$

30. $6p + 7q = 31$
$3p + 2q = 5$

31. $11s - 12t = 31$
$10s + 7t = 64$

32. $4b - 3h = 29$
$2b - h = 47$

33. $10r_1 + 3r_2 = 75$
$8r_1 - r_2 = 43$

34. $6\frac{2}{7}r + 3\frac{1}{7}r' = 110$
$9\frac{3}{7}r + 3\frac{1}{7}r' = 154$

35. $1.2x - y = 0.2$
$3x - 3\frac{1}{3}y = 4$

36. $3r + 4r' = 63$
$9r - 3r' = 54$

37. $8D + 3E = 77$
$7D - 2E = 35$

38. $T + M = 100$
$2T - 0.1M = 139.1$

173. Elimination by Substitution. It is often convenient to find from one equation the value of one unknown quantity in terms of the other, and to substitute this in the other equation.

1. Solve the system of equations

$$3x + 7y = 22.4 \tag{1}$$

$$x - 5y = 6 \tag{2}$$

From (2), $x = 6 + 5y$. (3)

Substituting in (1), $3(6 + 5y) + 7y = 22.4,$

or $18 + 15y + 7y = 22.4.$

Subtracting 18, $22y = 4.4.$

Dividing by 22, $y = 0.2.$

Substituting in (3), $x = 6 + 5 \times 0.2.$

$= 6 + 1 = 7.$

Check. $3 \times 7 + 7 \times 0.2 = 22.4,$

and $7 - 5 \times 0.2 = 6.$

This plan is sometimes advantageous when the coefficient of one of the unknown quantities is 1.

2. Solve the system of equations

$$5x + 2y = 34 \tag{1}$$

$$7x - 3y = 7 \tag{2}$$

From (1), $y = \frac{34 - 5x}{2}.$ (3)

Substituting in (2), $7x - 3 \cdot \frac{34 - 5x}{2} = 7.$

Multiplying by 2, $14x - 102 + 15x = 14.$

Solving, $x = 4.$

Substituting in (3), $y = \frac{34 - 5 \cdot 4}{2}$

$= 17 - 5 \cdot 2 = 7.$

3. Solve the system of equations

$$x + 3y = 7$$

$$x - 2y = 2$$

We have $x = 7 - 3y$ and $x = 2 + 2y$; hence $7 - 3y = 2 + 2y$, and $y = 1$. $\therefore x = 4$. This special form of elimination by substitution is sometimes called *elimination by comparison.*

174. Directions for Elimination. Therefore, to eliminate by substitution,

From one of the equations find the value of either unknown quantity in terms of the other.

Substitute this value in the other equation and solve.

Exercise 120. Elimination by Substitution

Examples 1 to 7, oral — Examples 8 to 21, written

1. If $x = 3$ and $x + y = 5$, what is the value of y?
2. If $y = 7$ and $x + y = 16$, what is the value of x?
3. If $x = -2$ and $x + y = 1$, what is the value of y?
4. If $y = -2$ and $x + y = 0$, what is the value of x?
5. If $r = 4$ and $r + r' = 9$, what is the value of r'?
6. If $r = 7$ and $r - r' = 5$, what is the value of r'?
7. If $P = 5$ and $P + Q = 9.8$, what is the value of Q?

Solve the following equations by substitution:

8. $x + 7y = 26$
   $2x + 3y = 19$
9. $x - 3y = -1$
   $3x + 2y = 19$
10. $x + 4y = 35$
    $3x - 2y = 7$
11. $x - 4y = 1$
    $5x + 2y = 49$
12. $x + 5y = 25$
    $7x + 3y = 47$
13. $x - 2y = 2$
    $6x + 5y = 80$
14. $2x + 17y = 61$
    $8x - y = 37$
15. $4.5x - 7y = 7$
    $5.5x + 6y = 59.5$
16. $0.8x + 0.3y = 11.3$
    $2x + 3.5y = 14.5$
17. $5m + 7n = 125$
    $7m - n = 13$
18. $5p - 3q = 27$
    $7q - 3p = 15$
19. $19x - y = 18$
    $27x + 4y = 31$
20. $F + 7M = 26$
    $F - 19M = -52$
21. $2E + 6F = 15.4$
    $E + 8.8F = 8.86$

Exercise 121. Problems

Examples 1 to 9, oral — Examples 10 to 36, written

1. Five more than three times a number is 20. What is the number?

2. The sum of two numbers is 34. One of the numbers is 19. What is the other number?

3. The difference of two numbers is 17. The larger number is 32. What is the smaller number?

4. The difference of two numbers is 41. The smaller number is 19. What is the larger number?

5. The difference of two numbers is 12. One of the numbers is 20. What is the other number? (Note that there are two possible answers.)

6. The product of two numbers is 51. One of the numbers is 17. What is the other number?

7. The quotient of two numbers is 27. The divisor is 3. What is the dividend?

8. The quotient of two numbers is 32. The dividend is 96. What is the divisor?

9. The quotient of two numbers is 10. One of the numbers is 20. What is the other number? (Note that there are two possible answers.)

10. The sum of two numbers is 32, and one of the numbers is three times the other. Find the numbers.

11. The sum of two numbers is 36, and one of the numbers is two more than the other. Find the numbers.

12. The sum of two numbers is 36, and one of the numbers is two less than the other. Find the numbers.

13. The sum of two numbers is 5.6, and one of the numbers is 4.2 more than the other. Find the numbers.

14. The sum of two numbers is 26.1, and one of the numbers is 11.96 more than the other. Find the numbers.

15. If one of two numbers is decreased by 1.3, the result is the other number. If three times the larger number is decreased by the smaller number, the result is 4.7. Find the numbers.

Let $x =$ the larger number,

and $y =$ the smaller number.

Then $$x - 1.3 = y, \qquad (1)$$

and $$3x - y = 4.7. \qquad (2)$$

Substituting the value of y from (1) in (2),

$$3x - x + 1.3 = 4.7.$$

$$\therefore\ 2x = 3.4,$$

and $$x = 1.7.$$

$$\therefore\ y = x - 1.3 = 1.7 - 1.3 = 0.4.$$

Therefore the numbers are 1.7 and 0.4.

Check. $1.7 - 1.3 = 0.4$, and $3 \times 1.7 - 0.4 = 4.7$.

16. If to twice one number we add three times a second number, the result is 14.7. If from twice the first number we subtract the second number, the result is 2.3. Find the numbers.

17. If to four times one number we add twice a second number, the result is 13½. If from five times the first number we subtract four times the second number, the result is 5½. Find the numbers.

18. The sum of two numbers is 11.74, and one of the numbers is 0.67 greater than the other. Find the numbers.

19. If to ten times one number we add five times a second number, the sum is 111. If to three times the first number we add seven times the second number, the sum is 55.3. Find the numbers.

20. The sum of two numbers is 55.6. Three times the second subtracted from twice the first is 74.7. Find the numbers.

21. The sum of two numbers is 19.6, and one of the numbers is one third the other. Find the numbers.

22. The difference of two numbers is 20.8, and one of the numbers is one fifth the other. Find the numbers.

23. If a rectangle were 2 in. longer and 3 in. wider, its area would be increased by 35 sq. in. If it were 2 in. shorter and 2 in. wider, the area would be unchanged. Find the dimensions.

Let $l =$ the number of inches of length,
and $w =$ the number of inches of width.
Then $lw =$ the number of square inches of area.
Then $(l+2)(w+3) = lw + 35,$
and $(l-2)(w+2) = lw.$
Simplifying, $3l + 2w = 29,$
and $l - w = 2.$
Solving, $l = 6.6,$
and $w = 4.6.$

Therefore the dimensions are 6.6 in. and 4.6 in.

Here the initial letters of length and width have been used to represent the unknown quantities, a custom that is coming into use. It is permissible to use x and y, l and w, or any convenient letters.

24. If a rectangle were 5 in. shorter and 1 in. wider, its area would be decreased by 35 sq. in. If it were 5 in. longer and 1 in. wider, its area would be increased by 65 sq. in. Find the dimensions.

25. If the perimeter of a rug is 20 ft., and if three times the length plus five times the width is 36 ft., what is the area?

26. The length of a room is $33\frac{1}{3}\%$ greater than the width, and the perimeter is 70 ft. Find the dimensions.

27. The width of a room is two thirds the length, and the length exceeds the width by 7 ft. Find the dimensions.

28. The length of a rectangular swimming tank is 50% greater than the width, and the perimeter is 160 ft. Find the dimensions.

29. The altitude of a rectangle is 20% less than the base, and the perimeter is 18 in. Find the dimensions.

30. The circumference of a circle is 10.708 in. greater than the diameter. Find the circumference and the diameter. (From § 15, $c = 3.1416\,d$.)

31. A bird flying with the wind travels 50 mi. an hour, but when flying against a wind that is twice as strong, it travels only 20 mi. an hour. Find the rate of the wind for each case. Find the rate of the bird in still air.

In such problems it is assumed that the rate of the wind should be added to the rate of the moving body when the latter goes with the wind, and subtracted when it goes against the wind. This is approximately the case in the problems considered.

Taking the rates to mean the number of miles per hour,

let $x =$ the rate of the bird,

and $y =$ the rate of the wind the first time.

Then $x + y = 50,$

and $x - 2y = 20.$

Solving, $x = 40,$

and $y = 10.$

$\therefore\ 2y = 20,$ the rate of the wind the second time.

32. An aëroplane flies with the wind at the rate of 75 mi. an hour, and against the wind at the rate of 45 mi. an hour. Find the rate of the wind. Find the rate of the aëroplane in still air.

33. A river steamer can go 26 mi. an hour with the current and 14 mi. an hour against it. Find the rate of the current. Find the rate of the steamer in still water.

34. The report of a gun was heard in 3 sec. at a place 3189 ft. distant, toward which the wind was blowing, and in 2 sec. at a place 2074 ft. distant, from which the wind was blowing. Find the velocity of sound and the rate at which the wind was blowing.

35. The report of a gun traveled $357\frac{1}{3}$ yd. a second with the wind and 346 yd. a second against the wind. Find the velocity of sound and the rate at which the wind was blowing.

36. A conductor walking from the rear to the front of a moving express train passes a telegraph pole at the rate of 45.2 mi. an hour. Walking to the rear of the train, he passes a telegraph pole at the rate of 40.6 mi. an hour. How fast is the train going, and how fast is the conductor walking?

175. Simultaneous Equations containing Fractions. If the equations contain fractions it is usually better to clear of fractions before attempting to eliminate.

Solve the equations

$$\frac{x+2}{x-2}=\frac{y+8}{y+4} \quad (1)$$

$$\frac{2x-5}{4y-1}=\frac{x-1}{2y+1} \quad (2)$$

Clearing (1), $xy+2y+4x+8=xy-2y+8x-16.$ (3)

Clearing (2), $4xy-10y+2x-5=4xy-4y-x+1.$ (4)

Simplifying (3), $x-y=6.$

Simplifying (4), $x-2y=2.$

Solving, $x=10,$

and $y=4.$

Check. $\frac{10+2}{10-2}=\frac{4+8}{4+4}=\frac{12}{8};$

$\frac{20-5}{16-1}=\frac{9}{9}.$

Exercise 122. Simultaneous Equations containing Fractions

Examples 1 to 5, oral — Examples 6 to 29, written

1 If $\frac{1}{x}=3$, what is the value of x?

2. If $\frac{3}{x}=7$, what is the value of x?

3. Find the value of x when $\frac{1}{x}=\frac{1}{7}$; when $\frac{2}{x}=\frac{3}{7}$. Check the results.

4. If $\frac{1}{x}+\frac{1}{y}=4$ and $\frac{1}{x}-\frac{1}{y}=2$, what is the value of $\frac{2}{x}$? of $\frac{1}{x}$? of x? Check the result.

5. If $\frac{1}{x}+\frac{2}{y}=4$ and $\frac{1}{x}-\frac{2}{y}=0$, what is the value of $\frac{2}{x}$? of $\frac{1}{x}$? of x? of y? Check the results.

Solve the following equations:

6. $$\frac{x}{3}+\frac{y}{4}=4$$
$$\frac{x}{4}+\frac{y}{3}=\frac{25}{6}$$

7. $$\frac{x+y}{3}=\frac{x-y}{5}$$
$$\frac{x-3y}{7}=2x+7$$

8. $$\frac{x-4}{x-3}=\frac{y+4}{y+7}$$
$$\frac{x+5}{x+2}=\frac{y-1}{y-2}$$

9. $$\frac{x+y}{8}+\frac{x-y}{6}=5$$
$$\frac{x+y}{4}-\frac{x-y}{3}=10$$

10. $$\frac{5}{x+2y}=\frac{7}{2x+y}$$
$$\frac{7}{3x-2}=\frac{5}{6-y}$$

11. $$\frac{3x-2}{5x-1}=\frac{3y+7}{5y+16}$$
$$\frac{3x-1}{x+5}=\frac{6y-5}{2y+3}$$

12. $$\frac{4x+5y}{40}=x-y$$
$$\frac{2x-y}{3}=\frac{1-4y}{2}$$

13. $$\frac{x+y}{3}+x=15$$
$$\frac{x-y}{5}+y=6$$

14. $$\frac{x+y}{7}+\frac{x-y}{2}=1$$
$$\frac{x+y}{2}-\frac{x-y}{8}=7\tfrac{1}{4}$$

15. $$\frac{x+2y+1}{2x-y+1}=2$$
$$\frac{3x-y+1}{x-y+3}=5$$

16. $$\frac{x+y}{3}+\frac{y-x}{2}=9$$
$$\frac{x}{2}+\frac{x+y}{9}=5$$

17. $$7x-\frac{y}{5}=48$$
$$\frac{35y+x}{7}=26$$

18. $$\frac{x+y-1}{x-y+1}=7$$
$$\frac{y-x+1}{x-y+1}=35$$

19. $$\frac{x+y+3}{x-y-3}=-\frac{3}{2}$$
$$\frac{x-y-3}{x-y+3}=-2$$

20. $$\frac{x-y}{x+y}=-\frac{8}{15}$$
$$\frac{63x-3y-44}{7}=100$$

21. $$\frac{x+3y+13}{4x+5y-25}=3$$
$$\frac{8x+y+6}{5x+3y-23}=5$$

22. Show that the set of equations $\frac{x+4}{x-2}=\frac{y+1}{y-1}$ and $\frac{3x-12}{3y-5}=\frac{2x+5}{2y+1}$ is indeterminate.

Solve the following equations:

23. $\frac{5x-2}{15-4y}=\frac{2.5x+3}{9-2y}$

$\frac{2x+3y+1}{4x+6y-8}=\frac{3x-7y+8}{6x-14y+6}$

24. $\frac{6x-7}{19-10y}=\frac{3x-4}{11-5y}$

$\frac{6x-10y-17}{3x-5y+2}=\frac{4x-14y-5}{2x-7y+12}$

25. $\frac{5(x-1.2y)}{13}+2=4y-3x$

$\frac{5(x-1.2y)}{6}+12=2y-\frac{2y-3x}{4}$

26. $8y-\frac{4(4+15y)}{3x-1}=\frac{8xy-53.5}{x+2.5}$

$2+3(3y+2x)=\frac{3(3y+2x)(3y-2x)+38}{1-2x+3y}$

27. $\frac{x+y}{3}+\frac{x-y}{3}=2.0944$

$\frac{3x+4y-1}{4}+\frac{2x-7y-3}{3}=2.7762$

28. $\frac{2.471x+4.11y-3}{4}-\frac{5.004x-3.012y}{3}=3.244$

$\frac{4.005x-3.034y+9}{4}+\frac{11.07x+1.233y-9}{3}=7.59$

29. $\frac{3.072x-4.002y+0.929}{2}+\frac{4.003x+8.066y+8}{4}=5.009$

$\frac{1.781x+3.024y+0.707}{4}+\frac{2.755x+2.208y+1.141}{5}=2$

Exercise 123. Problems

Examples 1 to 6, oral — Examples 7 to 47, written

1. Find the value of x, given: $\frac{1}{x}=7$; $\frac{1}{x}=1\frac{1}{2}$; $\frac{1}{x}=\frac{1}{5}$; $\frac{1}{x}=0.2$.

2. Find the value of k, given: $\frac{1}{k}=\frac{1}{3}$; $\frac{1}{k}=\frac{2}{3}$; $\frac{1}{k}=\frac{3}{4}$; $\frac{1}{k}=0.25$.

3. Find the value of r, given: $2\pi r=\pi$; $2\pi r=4\pi$; $2\pi r=\frac{1}{\pi}$.

4. Given $\frac{1}{x}+\frac{1}{y}=6$ and $\frac{1}{x}-\frac{1}{y}=4$, find the value of x.

5. The reciprocal of what number equals 9?

6. If one more than the reciprocal of a certain number equals $\frac{4}{3}$, what is the number?

7. If to the larger of two numbers we add 8, and then divide by the smaller number, the quotient is 2; but if from the larger we subtract 2, and then divide by the smaller, the quotient is 1. Find the numbers.

8. The larger of two numbers lacks one of being six times the smaller number; and if 2 is subtracted from the larger, and the remainder is divided by the smaller, the quotient is 5. Find the numbers.

9. A certain fraction $\frac{x}{y}$ becomes equal to $\frac{1}{2}$ if 3 is added to its numerator and 1 to its denominator, and equal to $\frac{1}{3}$ if 1 is added to its numerator and 1 is subtracted from its denominator. Find the fraction.

10. A certain fraction becomes equal to $\frac{2}{3}$ if 1 is added to its numerator and 2 is subtracted from its denominator, and equal to $\frac{1}{2}$ if 1 is subtracted from its numerator and 4 from its denominator. Find the fraction.

11. A certain fraction becomes equal to $\frac{1}{3}$ if 3 is subtracted from its numerator and from its denominator, and equal to $\frac{3}{5}$ if 1 is added to its numerator and to its denominator. Find the fraction.

12. A certain fraction becomes equal to $\frac{2}{3}$ if 7 is added to the numerator, and equal to $\frac{3}{8}$ if 7 is subtracted from the denominator. Find the fraction.

13. If the larger of two numbers is divided by the smaller, the quotient is 3 and the remainder 3; but if the smaller is divided by the larger, the quotient is 0.266 and the remainder 0.01. Find the numbers.

Let $x =$ the larger number,

and $y =$ the smaller number.

Then
$$\frac{x}{y} = 3 + \frac{3}{y}, \qquad (1)$$

and
$$\frac{y}{x} = 0.266 + \frac{0.01}{x}. \qquad (2)$$

Clearing (1) of fractions, $x - 3 = 3y$.

Clearing (2) of fractions,
$$y - 0.01 = 0.266x.$$

Eliminating y, $0.202x = 3.03$.

Dividing by 0.202, $x = 15$.

Substituting in (1), $\dfrac{15-3}{y} = 3$.

Solving, $y = 4$.

Therefore the numbers are 15 and 4.

14. If the larger of two numbers is divided by the smaller, the quotient is 3 and the remainder 3; but if the smaller is divided by the larger, the quotient is 0.2 and the remainder 2.6. Find the numbers.

15. If the smaller of two numbers is divided by the larger, the quotient is 0.8 and there is no remainder; but if the larger is divided by the smaller, the quotient is 1.2 and the remainder 0.36. Find the numbers.

16. If the larger of two numbers is divided by the smaller, the quotient is 4 and the remainder 9; but if twenty times the smaller is divided by twice the larger, the quotient is 2 and the remainder 152. Find the numbers.

17. A boatman rows 5 mi. down a river and back in 2 hr. He can row 1 mi. down the river in the same time that he can row 0.6 mi. up the river. Find the time he rows down and up respectively.

Let x = the number of hours going down,
and y = the number of hours going up.
Then $x + y = 2.$

Also, since the time to row a mile downstream is 0.6 of the time to row a mile upstream, and so for any other distance,

$$x = 0.6y.$$

Solving, $x = \frac{3}{4}$, and $y = 1\frac{1}{4}$.

18. A boat's crew can row down a river at the rate of 10 mi. an hour, and up the river at the rate of 5 mi. an hour. Find the rate of the stream, and their rate in still water.

19. A steam launch goes 20 mi. upstream and 36 mi. downstream in 8 hr. It can go 3 mi. downstream in the same time that it can go 1 mi. upstream. Find the rate of the stream, and the rate of the boat in still water.

Let b = the rate of the boat, in miles per hour,
and s = the rate of the stream.

Then $$\frac{20}{b-s} + \frac{36}{b+s} = 8,$$

and $$\frac{3}{b+s} = \frac{1}{b-s}.$$

Explain how these two equations are derived.
Solve the equations and check the results.

20. A steamer can go downstream at the rate of 22 mi. an hour, but its rate upstream is only $\frac{7}{11}$ as fast. Find the rate of the stream, and the rate of the steamer in still water.

21. Two bodies are 96 yd. apart. If they move toward each other with uniform (but unequal) rates, they will meet in 8 sec.; but if they move in the same direction, the swifter overtakes the slower in 48 sec. Find the rate of each.

22. When weighed in water, tin loses 0.137 of its weight, and copper 0.112 of its weight. If a 10-pound mass of tin and copper loses 1.195 lb., find the weight of the tin and the copper in the mass.

Let $t =$ the number of pounds of tin,

and $c =$ the number of pounds of copper.

Then $$t + c = 10, \quad (1)$$

and $$0.137t + 0.112c = 1.195. \quad (2)$$

Multiplying (1) by 0.112,

$$0.112t + 0.112c = 1.12.$$

Subtracting, $$0.025t = 0.075.$$

Dividing by 0.025, $$t = 3.$$

Whence $$c = 7.$$

Therefore there are 3 lb. of tin and 7 lb. of copper in the mass.

23. When weighed in water, silver loses 0.095 of its weight, and copper 0.112 of its weight. If a 12-pound mass of silver and copper loses 1.174 lb., find the weight of the silver and the copper in the mass.

24. When weighed in water, gold loses 0.051 of its weight, and silver 0.095 of its weight. If a 6-ounce piece of gold and silver loses 0.35 oz., find the weight of the gold and the silver in the piece.

25. When weighed in water, tin loses 0.137 of its weight, and lead 0.089 of its weight. If a 65-pound mass of tin and lead loses 6.025 lb., find the weight of the tin and the lead in the mass.

26. An iron bar covered with brass weighs 13 lb. When weighed in water, iron loses 0.128 of its weight, and brass 0.119 of its weight. The bar loses 1.655 lb. when weighed in water. Find the weight of the brass that covers the iron.

27. In water, 1 oz. of platinum weighs only 0.9535 oz., and 1 oz. of gold 0.949 oz. An ingot of gold and platinum that weighs 6 oz. in air weighs 5.6985 oz. in water. Find the weight of the gold and the platinum.

28. The sum of the digits in a number of two figures is 14. If the number of tens is increased by 4, and the number of units decreased by 4, the digits will be interchanged. Find the number.

Let $x =$ the tens' digit,
and $y =$ the units' digit.

Then the number is $10x + y$, just as 35 means 3 tens + 5, or $10 \times 3 + 5$.

Likewise, the number with the digits interchanged is $10y + x$, just as 35 becomes 53 when the digits are interchanged.

Then $$x + y = 14, \quad (1)$$
and $$10(x + 4) + (y - 4) = 10y + x. \quad (2)$$
Simplifying, $$x - y = -4.$$
Solving, $$x = 5, y = 9.$$

Therefore the number is 59.

If desired, we may let $t =$ the number of tens, and $u =$ the number of units.

29. The sum of the digits in a number of two figures is 6 and their difference is 4. What is the number? (Two answers.)

30. The sum of the digits in a number of two figures is 9. The number of units is half the number of tens. Find the number.

31. The sum of the digits in a number of two figures is 7. If 27 is subtracted from the number, the digits will be interchanged. Find the number.

32. The sum of the digits in a number of two figures is 12. The number of units is three times the number of tens. Find the number.

33. If a number of two digits is increased by 3, and the result divided by the sum of the digits of the number, the quotient is 4. If the number is divided by the number of its tens, the quotient is $12\frac{1}{3}$. Find the number.

34. The sum of the digits in a number of two figures is 8. If the number is divided by the sum of the digits, the quotient is 4.375. Find the number.

35. Twenty-seven coins, dollars and quarters, amount to $19.50. How many are there of each kind?

Let d = the number of dollars,
and q = the number of quarters.
Then $d + q = 27$,
and $d + \frac{q}{4} = 19.50$.

Explain the second equation and solve.

36. A man distributed $5.25 in the form of dimes and quarters among thirty boys, each boy receiving one coin. How many boys received dimes? How many received quarters?

37. A man distributed $3.50 in the form of dimes and nickels among fifty children, each child receiving one coin. How many received dimes? How many received nickels?

38. The admission to an entertainment was 50¢ for adults and 25¢ for children. The proceeds from 125 tickets were $51.25. How many adults were admitted? How many children?

39. A school gave an entertainment at which the tickets to pupils were sold at 40¢ each, and to others at 50¢ each. There were 245 tickets sold and the receipts were $108.50. How many of each kind were sold?

40. A grocer has in his cash drawer 106 bills, some one-dollar and the rest two-dollar bills. The total amount is $138. How many has he of each?

41. A receiving teller at a bank took in 515 bills, some five-dollar and the rest two-dollar bills. The total amount was $2155. How many of each did he receive?

42. A paymaster at a shop has 210 silver pieces, some quarters and the rest half dollars. The total amount is $75. How many has he of each?

43. A dealer has 19 pieces of iron pipe, some 12 ft. long and the rest 6 ft. The total length of the pipe is 168 ft. How many pieces has he of each?

44. In running a quarter-mile race A gives B a start of 32 yd. and beats him by 1 sec. In a second trial A gives B a start of 6 sec. and is beaten by 8.8 yd. Find the rate of each.

Let $a =$ A's rate in yards per second,

and $b =$ B's rate in yards per second.

Since $\frac{1}{4}$ mi. $= 440$ yd.,

therefore $\frac{440}{a} =$ number of seconds A takes for $\frac{1}{4}$ mi.

Since B has a start of 32 yd., he runs 408 yd. the first trial;

and $\frac{408}{b} =$ number of seconds B was running.

But B took 1 sec. longer than A,

therefore $\frac{440}{a} + 1 =$ number of seconds B was running.

$$\therefore \frac{440}{a} + 1 = \frac{408}{b}.$$

In the second trial A has run $(440 - 8.8)$ yd., or 431.2 yd., when B hits the tape.

$$\therefore \frac{440}{b} = \frac{431.2}{a} + 6.$$

Solving, $a = 8.8$, $b = 8$.

Therefore A runs 8.8 yd. a second, and B runs 8 yd. a second.

45. In running a half-mile race A gives B a start of 56 yd. and beats him by 3 sec. In a second trial A gives B a start of 30 sec. and is beaten by 88 yd. Find the rate of each per second.

46. In running a half-mile race A beats B by 4.4 sec. In a second trial A gives B a start of 25 yd. and beats him by 2.5 yd. Find the rate of each per second, and the time required by each to run a mile.

47. A and B run a mile race, A winning by 6 sec. In a second trial A gives B a start of 40 yd. and is beaten by half a second. Find the rate of each per second, and the time required by each to run a mile.

176. Literal Simultaneous Equations. Although it is coming to be a custom to use initial letters for the unknown quantities in numerical equations, it is still common to employ the first letters of the alphabet for numbers supposed to be known, and the last letters for those supposed not to be known, in literal equations.

Solve the equations

$$ax + by = c, \tag{1}$$

$$a'x + b'y = c'. \tag{2}$$

Here a and a' are entirely different, it being convenient to use similar forms as coefficients of the same letter x.

Multiplying (1) by b', $\quad ab'x + bb'y = b'c.$

Multiplying (2) by b, $\quad \underline{a'bx + bb'y = bc'.}$

Subtracting, $\quad (ab' - a'b)\,x = b'c - bc'.$

Dividing by $ab' - a'b$, $\quad x = \dfrac{b'c - bc'}{ab' - a'b}.$

It now remains to find the value of y. This may be done as in the case of x, or we may substitute the value of x in one of the given equations. Substituting in (1), we have

$$a \cdot \frac{b'c - bc'}{ab' - a'b} + by = c.$$

Subtracting $a \cdot \dfrac{b'c - bc'}{ab' - a'b}$,

$$\begin{aligned} by &= c - a \cdot \frac{b'c - bc'}{ab' - a'b} \\ &= \frac{ab'c - a'bc - ab'c + abc'}{ab' - a'b} \\ &= \frac{b\,(ac' - a'c)}{ab' - a'b}. \end{aligned}$$

Dividing by b, $\quad y = \dfrac{ac' - a'c}{ab' - a'b}.$

We might have found y by noticing that if we interchanged x and y in the equations we had to interchange a and b and also a' and b'. We may therefore write the result for y by making these changes in the value of x.

The teacher should use his discretion as to the amount of checking to be required in the case of literal equations. While the only complete check is that of substitution in the original equations, this is often so tedious that the teacher may prefer to tell whether the result is correct.

Exercise 124. Literal Simultaneous Equations

Examples 1 to 6, oral — Examples 7 to 21, written

1. Solve the equations: $x + y = a,\ x - y = b.$

2. Solve the equations: $x + y = 3a,\ x - y = a.$

3. Solve the equations: $2x + y = a,\ 2x - y = b.$

4. Solve the equations: $2x + y = 3a,\ 2x - y = a.$

5. Solve the equations: $x + y = 4a,\ y = a.$

6. Solve the equations: $x+y=2a+b,\ x-y=2a-b.$

Solve the following equations:

7. $$x + y = 15a + b \qquad x - y = 15a - b$$

8. $$ax + y = m \qquad bx - y = n$$

9. $$ax + by = k \qquad cx - dy = l$$

10. $$a^2x + b^2y = c^2 \qquad p^2x + q^2y = r^2$$

11. $$x + y = \frac{2(a^2 + b^2)}{a^2 - b^2} \qquad x - y = \frac{4ab}{a^2 - b^2}$$

12. $$\frac{x + y + 1}{x + y - 1} = m \qquad \frac{x - y + 1}{x - y - 1} = n$$

13. $$\frac{x - a + c}{y - a + b} = \frac{b}{c} \qquad \frac{x + c - a}{y + a - b} = \frac{c}{b}$$

14. $$ax + by = 3ab \qquad a^2x + b^2y = a + b$$

15. $$ax + by = c \qquad bx + ay = d$$

16. $$ax - by = cd \qquad bx - ay = ef$$

17. $$abx + cdy = k \qquad pqx + mny = l$$

18. $$\frac{x + y}{x - y} = \frac{a}{b - c} \qquad \frac{x + c}{y + b} = \frac{a + b}{a + c}$$

19. $$\frac{x + 1}{x - 1} + \frac{y - 1}{y + 1} = 3 \qquad \frac{x + 1}{x - 1} - \frac{y - 1}{y + 1} = 3$$

20. $$\frac{x + a}{y + b} = \frac{a + b + c}{a - b + c} \qquad \frac{x - a}{y - b} = \frac{a + b - c}{a - b + c}$$

21. $$(a + b + c + d)x + (a - b + c + d)y = a + b - c + d$$
$$(a - b - c + d)x - (a - b + c + d)y = a - b + c + d$$

177. Special Forms of Fractional Equations. In the case of simultaneous fractional equations it is often advisable to eliminate without clearing of fractions.

Solve the equations

$$\frac{3}{x}+\frac{4}{y}=1 \qquad (1)$$

$$\frac{21}{x}-\frac{2}{y}=2 \qquad (2)$$

Multiplying (2) by 2, $\frac{42}{x}-\frac{4}{y}=4.$

Adding (1), $\frac{45}{x}=5.$

Multiplying by x and dividing by 5,

$$9=x.$$

Substituting 9 for x in (1), $\frac{3}{9}+\frac{4}{y}=1,$

or $\frac{4}{y}=\frac{2}{3}.$

Solving for y, $y=6.$

Exercise 125. Fractional Simultaneous Equations

Examples 1 to 4, oral — Examples 5 to 19, written

1. When $\frac{a}{x}+\frac{b}{y}=4$, and $\frac{a}{x}-\frac{b}{y}=2$, what is the value of $\frac{2a}{x}$? of $\frac{a}{x}$? of a in terms of x? of x in terms of a?

2. When $\frac{a}{x}+\frac{b}{y}=4$, and $-\frac{a}{x}+\frac{b}{y}=2$, what is the value of $\frac{2b}{y}$? of $\frac{b}{y}$? of y?

3. When $\frac{1}{x}+\frac{1}{y}=6$, and $\frac{1}{x}-\frac{1}{y}=2$, what is the value of $\frac{2}{x}$? of $\frac{1}{x}$? of x? How will you find the value of y?

4. When $\frac{1}{x}+\frac{1}{y}=a$, and $\frac{1}{x}-\frac{1}{y}=b$, what is the value of $\frac{2}{x}$? of $\frac{1}{x}$? How will you find the value of x?

Solve the following equations:

5. $\frac{3}{x}+\frac{4}{y}=3$
$\frac{9}{x}-\frac{8}{y}=-1$

6. $\frac{15}{x}+\frac{6}{y}=8$
$\frac{21}{x}-\frac{12}{y}=1$

7. $\frac{x}{3}+\frac{y}{4}=6$
$\frac{x}{6}-\frac{y}{3}=-2\frac{1}{2}$

8. $\frac{15}{2x}+\frac{1}{y}=2$
$\frac{25}{2x}-\frac{3}{y}=1$

9. $\frac{x}{5}-\frac{y}{10}=0.75$
$\frac{x}{3}+\frac{y}{5}=2\frac{1}{6}$

10. $\frac{43.2}{5x}-\frac{6.2}{5y}=\frac{4}{5}$
$\frac{64.8}{5x}+\frac{12.4}{5y}=2.6$

11. $\frac{9.8}{7x}+\frac{10.5}{5y}=2$
$\frac{29.4}{7x}-\frac{6.3}{y}=1$

12. $\frac{1}{2x}+\frac{1}{y}=p$
$\frac{1}{2x}-\frac{1}{y}=q$

13. $\frac{a}{x}+\frac{b}{y}=c$
$\frac{b}{x}+\frac{a}{y}=d$

14. $\frac{1}{ax}+\frac{1}{by}=c$
$\frac{1}{bx}-\frac{1}{ay}=d$

15. $\frac{a}{bx}+\frac{b}{ay}=a+b$
$\frac{b}{x}+\frac{a}{y}=a^2+b^2$

16. $\frac{a}{x}+\frac{b}{y}=c$
$\frac{a'}{x}+\frac{b'}{y}=c'$

17. $\frac{a+b}{x}+\frac{a-b}{y}=a^2$
$\frac{a-b}{x}-\frac{a-b}{y}=a$

18. $\frac{a}{a+x}+\frac{b}{b+y}=c$
$\frac{b}{a+x}+\frac{a}{b+y}=d$

19. $4(x-3)+\frac{3}{4}x+7+\frac{2}{3}y=\frac{1}{4}x+15+\frac{1}{6}y+4x-12$
$\frac{2}{3}(6x+15)+\frac{1}{2}x+3(3+\frac{1}{4}y)=x+9+4x+\frac{1}{3}(y+9)+10$

178. Three Simultaneous Equations. If three simultaneous equations are given, involving three unknown quantities, the quantities are eliminated by combining pairs of equations, as shown in the following solution.

Solve the equations

$$5x + 2y - 4z = -3 \qquad (1)$$

$$3x - 3y + 5z = 12 \qquad (2)$$

$$4x + 5y + 2z = 20 \qquad (3)$$

We may eliminate z by combining (1) and (2), as follows:

Multiplying (1) by 5, $25x + 10y - 20z = -15.$ (4)

Multiplying (2) by 4, $12x - 12y + 20z = 48.$ (5)

Adding, $37x - 2y = 33.$ (6)

We may eliminate z between (1) and (3), as follows:

Multiplying (3) by 2, $8x + 10y + 4z = 40.$ (7)

Adding (1) and (7), $13x + 12y = 37.$ (8)

We now have two equations, (6) and (8), involving x and y. We may now eliminate y as follows:

Multiplying (6) by 6, $222x - 12y = 198.$ (9)

Adding (8) and (9), $235x = 235.$

$\therefore x = 1.$

Substituting in (8), $13 + 12y = 37.$

Hence $12y = 24,$

and $y = 2.$

Substituting $x = 1$, $y = 2$, in (1), $5 + 4 - 4z = -3.$

Solving for z, $z = 3.$

Therefore $x = 1,\ y = 2,\ z = 3.$

Check. Substituting in (1), (2), and (3), we have

$$5 + 4 - 12 = -3,$$

$$3 - 6 + 15 = 12,$$

$$4 + 10 + 6 = 20.$$

In checking, it is usually sufficient to substitute in the equation which has been used least in the solution. In this case the equation is (3). To be sure of the results, however, it is necessary to substitute in all three of the original equations.

Exercise 126. Three Simultaneous Equations

Examples 1 to 4, oral — Examples 5 to 54, written

1. In the system of equations

$$x + y + z = 6 \tag{1}$$

$$x + y - z = 4 \tag{2}$$

$$x - y + z = 2 \tag{3}$$

eliminate y and z at the same time from (2) and (3). What is the value of x?

2. Eliminate z from (1) and (2) and find the value of $x + y$. Then substitute the value of x found in Ex. 1, and find the value of y.

3. Substitute the values of x and y found in Exs. 1 and 2, and find the value of z from (1).

4. Check the values of x, y, and z found in Exs. 1–3, by substituting in (2).

Solve the following equations:

5. $$\begin{aligned} x + y + z &= 10 \\ x - y + z &= 2 \\ x + y - z &= 8 \end{aligned}$$

6. $$\begin{aligned} x + y + z &= 7 \\ 3x + y - z &= 3 \\ 2x + 4y + z &= 12 \end{aligned}$$

7. $$\begin{aligned} x + y + z &= 13 \\ 3x + y - 3z &= 5 \\ x - 2y + 4z &= 10 \end{aligned}$$

8. $$\begin{aligned} x + 2y + 3z &= 41 \\ x - 3y + 4z &= 6 \\ 5x + 6y - 7z &= 63 \end{aligned}$$

9. $$\begin{aligned} 2x - y + z &= 3 \\ x + 2y + z &= 12 \\ 4x - 3y + z &= 1 \end{aligned}$$

10. $$\begin{aligned} 3x + 2y - 4z &= 15 \\ 5x - 3y + 2z &= 60 \\ 2x + 4y - 3z &= 45 \end{aligned}$$

11. $$\begin{aligned} x - 3y + z &= 10 \\ 2x - 7y - 5z &= -2 \\ x + y - 2z &= 5 \end{aligned}$$

12. $$\begin{aligned} 10x + 8y - 9z &= 10 \\ 12x + 2y - 16z &= 15 \\ 2x + 10y - 25z &= 0 \end{aligned}$$

13. $$\begin{aligned} x + y + z &= 14.6 \\ x - y + z &= 3.4 \\ x + y - z &= 12.2 \end{aligned}$$

14. $$\begin{aligned} x + 2y - z &= 3.25 \\ 3x - y + z &= 3 \\ x + y - 5z &= 1.92 \end{aligned}$$

Solve the following equations:

15. $x - y = 3$
$y - z = 5$
$z + x = 9$

16. $x + y = 2$
$y + z = -2$
$z + x = 12$

17. $x + y = 5$
$y + z = 3$
$z + x = 7$

18. $2y + z = 9$
$z - 2y = 1$
$x + y + z = 1$

19. $x + 3 = 5 - 4y$
$x + z = 3y$
$8y - 4 = z$

20. $x + y + z = 7.77$
$3x + y - z = 6.51$
$7x - 2y + 3z = -11.49$

21. $\frac{x}{b} + \frac{y}{c} + \frac{z}{a} = 3$
$\frac{x}{b} + \frac{y}{c} - \frac{z}{a} = 1$
$\frac{x}{b} - \frac{y}{c} + \frac{z}{a} = 1$

22. $\frac{x}{6} + \frac{y}{3} + \frac{z}{4} = 36$
$\frac{x}{9} + \frac{y}{15} + \frac{z}{20} = 10$
$\frac{x}{4} + \frac{y}{2} + \frac{z}{10} = 43$

23. $ax + by - cz = 2ab$
$by + cz - ax = 2bc$
$cz + ax - by = 2ac$

24. $3(z - 1) = 2(y - 1)$
$4(x + y) = 9z - 4$
$2y - 9 = 7(5x - 3z)$

25. $0.7x - 1.1y + 0.2z = 1$
$x - 0.3y - 0.5z = 1.5$
$1.2x - 0.1y - 0.6z = 3.1$

26. $(a + b)x + (a - b)z = 2bc$
$(b + c)y + (a + b)x = 2ac$
$(a - b)z + (b + c)y = 2ab$

27. $ax - by + cz = a^2$
$ax + by - cz = b^2$
$-ax + by + cz = c^2$

28. $ax + by + cz = a$
$ax - by - cz = b$
$ax + cy + bz = c$

29. $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 12$
$\frac{2}{x} - \frac{3}{y} + \frac{4}{z} = 14$
$\frac{3}{x} + \frac{5}{y} - \frac{7}{z} = -6$

30. $\frac{2}{x} - \frac{3}{y} + \frac{4}{z} = 2.9$
$\frac{5}{x} - \frac{6}{y} - \frac{7}{z} = -10.4$
$\frac{9}{y} + \frac{10}{z} - \frac{8}{x} = 14.9$

31. A and B can do a piece of work in 3 da., B and C in 4 da., C and A in 5 da. How long will it take each to do the work?

Let a, b, c = the number of days required by A, B, C, respectively.

Then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ = the parts they can do in 1 da., respectively.

Then $\frac{1}{a} + \frac{1}{b}$ = the part A and B together can do in 1 da.

But $\frac{1}{3}$ = the part A and B together can do in 1 da.

Therefore
$$\frac{1}{a} + \frac{1}{b} = \frac{1}{3}. \tag{1}$$

Similarly,
$$\frac{1}{b} + \frac{1}{c} = \frac{1}{4}, \tag{2}$$

and
$$\frac{1}{c} + \frac{1}{a} = \frac{1}{5}. \tag{3}$$

Adding and dividing by 2,
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{47}{120}.$$

Subtracting (1),
$$\frac{1}{c} = \frac{7}{120}.$$

Similarly,
$$\frac{1}{b} = \frac{23}{120},$$

and
$$\frac{1}{a} = \frac{17}{120}.$$

Solving,
$$a = 7\tfrac{1}{17},\ b = 5\tfrac{5}{23},\ c = 17\tfrac{1}{7}.$$

Therefore it will take A $7\frac{1}{17}$ da., B $5\frac{5}{23}$ da., and C $17\frac{1}{7}$ da.

32. A and B can do a piece of work in 7 da., B and C in 6 da., C and A in 5 da. How long will it take each to do the work?

33. A cistern can be filled by two pipes, X and Y, in 35 min., by X and Z in 42 min., and by Y and Z in 70 min. How long will it take X, Y, and Z to fill it? How long will it take each?

34. A and B can build a wall in p days, B and C in q days, C and A in r days. How long will it take them working together to build the wall?

35. A and B can remove a pile of bricks in 2½ da., B and C in 4 da., C and A in 3⅓ da. How long will it take them working together to remove the pile? How long will it take each working alone?

36. Three villages, A, B, and C, are situated at the vertices of a triangle. The distance from A to B by way of C is 76 mi.; from A to C by way of B, 79 mi.; from B to C by way of A, 81 mi. Find the direct distance from A to B; from B to C; from C to A.

37. Three boys were playing marbles, when A remarked that if B gave him one of his marbles, B would have twice what A then had. C remarked that if B gave him three marbles, C would have twice what B then had. A then remarked that if C gave him seven of his marbles, the number that A would have would lack three of being half as many as C would have left. How many marbles did each have?

38. A printing office furnishes 1000 cards, 2000 billheads, and 3000 letterheads for $10.50; 2000 cards, 1000 billheads, and 2000 letterheads for $8.50; 1000 cards, 1000 billheads, and 2000 letterheads for $7. Find the cost of each per thousand.

39. In an athletic meet the following was the final score of teams A, B, and C:

	1st Places	*2d Places*	*3d Places*	*Total No. of Points*
A	5	3	2	36
B	2	4	1	23
C	2	2	6	22

How many points did each place count?

The first equation is $5x + 3y + 2z = 36$.

40. A cheese factory received 69,000 lb. of milk in June and July, 53,000 lb. in July and August, and 94,000 lb. in June, July, and August. How many pounds did it receive in each of the months?

41. The sum of the three digits of a number is 8. The digit in the units' place exceeds that in the tens' by 3, and if 396 is added to the number the order of the digits will be reversed. Find the number.

Let $h =$ the hundreds' digit,

$t =$ the tens' digit,

and $u =$ the units' digit.

Then $100h + 10t + u =$ the number.

But $$h + t + u = 8, \quad (1)$$

$$u - t = 3, \quad (2)$$

and $$100h + 10t + u + 396 = 100u + 10t + h, \quad (3)$$

since the order of the digits is now reversed.

From (1) and (2), $h + 2u = 11.$

From (3), $99h - 99u = -396,$

whence $h - u = -4.$

Eliminating h, $u = 5.$

From (2), $t = u - 3 = 5 - 3 = 2.$

From (1), $h = 8 - t - u = 8 - 5 - 2 = 1.$

Therefore the digits are 1, 2, 5, and the number is 125.

42. The sum of the three digits of a number is 12. The sum of the hundreds' and tens' digits is four more than the units' digit. The sum of the hundreds' and units' digits is four more than the tens' digit. Find the number.

43. The sum of the three digits of a number is 9. The sum of the hundreds' and units' digits is 6, and if 198 is added to the number the order of the digits will be reversed. Find the number.

44. The sum of the three digits of a number is 6. The sum of the hundreds' and tens' digits is 3, and the units' digit is 3 more than the tens' digit. Find the number.

45. The sum of the three digits of a number is 18. The value of the number is not changed if the order of the digits is reversed, and the sum of the hundreds' and units' digits equals the sum of the units' and tens' digits. Find the number.

46. In any triangle the sum of the three angles is 180°. In a certain triangle *ABC* the sum of angles *A* and *B* is 130°, and the sum of angles *B* and *C* is 110°. Find the number of degrees in each of the three angles.

47. In Ex. 46, if angle *A* is 30° larger than angle *B*, and angle *B* is 30° larger than angle *C*, find the angles.

48. In a certain triangle *ABC* angle *A* is 100° larger than angle *B* and 110° larger than angle *C*. Find the number of degrees in each of the three angles.

49. Of the three angles of a triangle, the sum of the second and third equals twice the first, and the difference between the second and first equals five times the third. Find the number of degrees in each of the three angles.

50. The cost of repairs in a sawmill was $\frac{3}{16}$ of the total expenses for the month, and the total expenses were $\frac{3}{4}$ of the gross earnings. The net proceeds for the month were $256 Required the gross earnings and the running expenses.

51. A food ration for a horse contains $\frac{4}{5}$ as much oats as corn and $\frac{1}{3}$ as much mineral-carrying food as other kinds of food. How much of each kind of food is there in 20 lb. of mixed feed?

52. Three postmen deliver mail on routes of different lengths. The first route is 1 mi. shorter than the second and $1\frac{2}{3}$ mi. shorter than the third. The average length of the routes is $4\frac{8}{9}$ mi. Find the length of each route.

53. Three boys went fishing, and on their return they gave this problem: "We caught 19 fish in all, and one of us caught four more than either of the others. How many did each of us catch?"

54. If x, y, and z are three numbers such that $3x = y + 13.6$, $3y = 2z + 6.4$, and twice the sum of the numbers, plus the second, plus twice the third is 67.8, what are the numbers?

179. Four or More Simultaneous Equations. Four simultaneous equations may be solved by eliminating one unknown quantity from one pair of equations, the same quantity from another pair, and so on until three equations involving three unknown quantities result, these being solved as usual.

Thus we may eliminate some letter like z from the first and second equations, then from the first and third, then from the first and fourth. We then have three equations with three unknown quantities.

Exercise 127. Four or More Simultaneous Equations

Examples 1 and 2, oral — Examples 3 to 9, written

1. In Ex. 3 which letter will you eliminate first? Which equations will you use for this purpose? Why?

2. Answer the same questions for Ex. 4; for Ex. 5; for Ex. 6.

Solve the following equations:

3. $w+x+y+z=10$
$w+x+y-z=2$
$w+x-y+z=4$
$w-x+y+z=6$

4. $w+x+y+z=18$
$w-2x+y+2z=12$
$w+3x-2y-z=-1$
$4w-x+3y-5z=-7$

5. $w+2x+3y+4z=3$
$2w+3x+4y+5z=3$
$6w+7x-8y+9z=35$
$64w+16x+4y+z=-54$

6. $w+x+y=6$
$x+y+z=6$
$y+z+w=6$
$z+w+x=6$

7. $w+2x+3y=12$
$w+4x-3y=14$
$w+3x+y=14$
$x+y+z=9$

8. $u+v+x=7$
$2v+2x+2y=8$
$3x+3y+3u=24$
$4y+4u+4v=28$

9. A number consists of four digits. The sum of the hundreds', tens', and units' digits is 9; of the hundreds', tens', and thousands' digits, 6; of the tens', units', and thousands' digits, 8; of the units', thousands', and hundreds' digits, 7. What is the number?

Exercise 128. Miscellaneous Problems

Examples 1 to 6, oral — Examples 7 to 32, written

1. In a certain number the sum of the units' and tens' digits is 6, and the tens' digit is half the units' digit. State the equations.

2. The sum of two numbers is 10 and their difference is 6. State the equations.

3. Twice the first of two numbers added to the second equals 9. Twice the second added to the first equals 6. State the equations.

4. If a man can do a piece of work in d days, what part of the work can he do in one day?

5. How will you represent a general number of two digits? of three digits?

6. If A can do a piece of work in a days, and B can do it in b days, what part of the work can they do in one day, working together?

7. A pupil in the arithmetic class was told to add 3 to a certain number and divide the sum by 2. Misunderstanding the problem he subtracted 2 from the number and multiplied by 3, and yet he obtained the correct result. What was the number?

8. A pupil was told to add a to a certain number and divide the sum by b. Misunderstanding the problem he subtracted b from the number and multiplied by a, and yet he obtained the correct result. What was the number? Evaluate the result for $a = 3$, $b = 2$; also for $a = 3$, $b = 1$.

9. Two trains go from A to B over different roads, one of which is 15 mi. longer than the other. The train on the shorter route takes 6 hr., and the one on the longer route, traveling 10 mi. less per hour, takes 8 hr. 30 min. Find the length of each route

10. A boy riding a bicycle at the rate of 9 mi. an hour is sent to overtake a boy who is riding horseback at the rate of 6 mi. an hour and who had 4 mi. the start. How long will it take the second boy to overtake the first?

11. The cost of publication of each copy of a certain illustrated magazine is $6\frac{1}{2}$¢. It sells to dealers for 6¢, and the amount received for advertising is 10% of the amount received for all the magazines in excess of 10,000. Find the least number of magazines that can be issued without loss.

12. A man bought 10 cows and 50 sheep for $750. He sold the cows at a profit of 10%, and the sheep at a profit of 30%, receiving in all $875. Find the average cost of the cows and the average cost of the sheep.

13. A number of boys purchase a camp. If there had been two more in the company, each would have paid $12 less; and if there had been three less, each would have paid $24 more. How many boys were there and how much did each of them pay?

14. A boat's crew can row 15 mi. an hour downstream. The crew can row a certain distance in still water in 15 min., and requires 20 min. to row the same distance upstream. Find the rate of the stream and the rate of rowing in still water.

15. The perimeter of a rectangle is 60 ft. If the length is increased by 3 ft. and the width is decreased by 3 ft., the area is decreased by 21 sq. ft. Find the dimensions.

16. The length of a rectangle is twice its width. If the length and width are both increased by 1 in., the area is increased by 31 sq. in. Find the dimensions.

17. If the length of a rectangular rug is increased by 5 ft. and the width is decreased by 2 ft., the area is increased by 10 sq. ft. If the length is increased by 2 ft. and the width is decreased by 5 ft., the area is decreased by 65 sq. ft. Find the dimensions of the rug.

18. In the equation $B = xA + y$, it is known that when $B = 18$ the value of A is 8, and when $B = 46$ the value of A is 16. Find the values of x and y.

19. In the equation $F = aB + c$, it is known that when $F = 110$ the value of B is 8, and when $F = 210$ the value of B is 16. Find the values of a and c.

20. In the equation $R = aE + b$, it is known that when $R = 40$ the value of E is 10, and when $R = 220$ the value of E is 50. Find the values of a and b.

21. Of two squares of carpet, the perimeter of one is 44 ft. more than that of the other, and the area of the one is 187 sq. ft. more than the area of the other. Find the side of each.

22. Two pictures are framed in the same manner. The first is 1 ft. 6 in. by 2 ft., and the second is 2 ft. by 2 ft. 6 in. The frame, glass, and labor for the first cost $1.50, of which 36¢ was for labor and the extra molding for the corners; and that for the second cost $2.10, of which 42¢ was for labor and the corners. What is the price of the glass per square foot, and the price of the frame per linear foot?

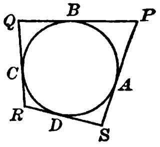
23. The dimensions of a rug are such that if the length were $1\frac{1}{2}$ ft. less, the rug would be a square; but if the width were $5\frac{1}{2}$ in. less, the length would be double the width. Find the dimensions.

24. A lady paid 76¢ for some sugar and tea. The sugar cost 6¢ a pound and the tea 80¢ a pound. The sugar weighed twelve times as much as the tea. How many pounds of each did she buy?

25. A lady bought 12 lb. of sugar, 2 lb. of tea, and 5 doz. eggs, paying $3.74 for all. The tea cost ten times as much per pound as the sugar. If the tea had cost 10¢ less per pound she could have bought 2 doz. eggs for the cost of 1 lb. of tea. Find the price of the sugar and tea per pound, and of the eggs per dozen.

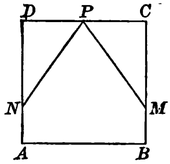
26. In any quadrilateral the sum of the four angles equals 360°. There is a quadrilateral with angles A, B, C, and D, such that $A + B + C = 280°$, $B + C + D = 280°$, and $C + D = 190°$. Find the size of each angle.

27. In this figure the following equalities are known: $PA = PB$, $QB = QC$, $RC = RD$, $SA = SD$. It is further known that $PS = 9$, $PQ = 8$, $QR = 5$, and $PA = SA + 1$. Find the lengths of SA, AP, PB, BQ, QC, CR, RD, DS.

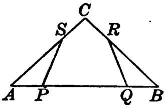


28. A box contains a mixture of 6 qt. of oats and 9 qt. of corn, and another box contains a mixture of 6 qt. of oats and 2 qt. of corn. How many quarts must be taken from each box to have a mixture of 7 qt., half of which will be oats and half corn?

29. In this square M, P, and N are so taken that $BM = \frac{1}{2} MC$, and $CP = PD$. The perimeter of the square is 40. Find the lengths of BM, MC, and CP.



30. In this triangle $AC = BC$ and $AP = QB = RC = CS$. It is also given that $AQ = 3\frac{1}{2}$, $BC = 3$, and the perimeter of the triangle is $10\frac{1}{2}$. Find the lengths of AP, PQ, and BR.



31. A person has $18,375 to invest. He can buy 3% bonds at 75 (a $100 bond that pays 3% interest being purchased by him for $75), and 5% bonds at 120. How much of his money must he invest in each kind of bond to have the same income from each investment?

32. The formula for the area of a trapezoid is $a = \frac{1}{2} h (b + b')$, where h is the height and b and b' are the bases. It is known that a certain trapezoid with a height of 10 in. has an area of 140 sq. in., and that the upper base of the trapezoid is 4 in. shorter than the lower base. Find the length of each base of the trapezoid.

180. Discussion of a Problem. When the result of a problem is expressed in letters as a general formula, the interpretation of the result is called a *discussion of the problem.*

An express train and a freight train are traveling along parallel tracks in the same direction. The express travels e miles an hour and the freight f miles an hour. At noon the freight is d miles in advance of the express. When will the express pass the freight?

If the express passes the freight x hours after noon, it will have traveled ex miles, while the freight will have traveled fx miles. And since the express must gain d miles, we have

$$ex = fx + d;$$

whence $$x = \frac{d}{e-f}.$$

1. *Suppose e is greater than f* (expressed thus: $e > f$).

Then the denominator is positive and x is positive. This is as it should be, since the express is then traveling faster than the freight and will overtake it *after* noon.

2. *Suppose e is less than f* (expressed thus: $e < f$).

Then x is negative. This is as it should be, since the express is traveling slower than the freight. Therefore if the trains were together at any time it was *before* noon.

3. *Suppose $e = f$.*

Then $x = \frac{d}{0}$, an expression excluded by § 74. This fraction may, however, be regarded as a *symbol of infinity* (∞), and written $\frac{d}{0} = \infty$. For if the trains are d miles apart and are traveling at the same rate, they can never be together.

4. *Suppose $e = f$, and $d = 0$.*

Then $x = \frac{0}{0}$. But if the trains are traveling at the same rate and are no distance apart, they are always together. Therefore $\frac{0}{0}$ is to be regarded as a *symbol of indetermination.*

Exercise 129. Discussion of Problems

Examples 1 to 3, oral — Examples 4 to 11, written

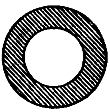
1. If $x = \frac{a}{b-c}$, and $b > c$, is x positive or is it negative?

2. In Ex. 1 suppose $b < c$; suppose $b = c$.

3. In Ex. 1 what interpretation is to be given to the result when $a = 0$ and $b = c$?

4. A train traveling r miles an hour is t hours ahead of a second train that travels r' miles an hour. In how many hours will the second train overtake the first? Discuss the problem when $r > r'$; when $r = r'$; when $r < r'$.

5. In the figure here shown, the area of the outer circle is πr_1^2 and that of the inner circle is πr_2^2. Hence the area of the shaded ring is given by the formula $a = \pi(r_1^2 - r_2^2) = \pi(r_1 + r_2)(r_1 - r_2)$. Discuss the problem when $r_1 > r_2$; when $r_1 = r_2$.



6. If a is divided by $b - c$, the result is q. Discuss the nature of q when $b > c$; when $b < c$; when $b = c$; when $a = 0$ and $b = c$; when $a = 0$ and $b > c$ or $b < c$.

7. Two boys live d miles apart. They start from their respective homes at the same time and walk toward each other, one at the rate of a miles an hour, and the other at the rate of b miles an hour. In how many hours after starting will they meet? Discuss the problem when $d = a = b$; when $d = a$ and $b = 0$; when $d = 0$.

8. Discuss the formula $a - b = \frac{a^2 - b^2}{a + b}$ when $a > b$; when $a < b$; when $a = b$.

9. Discuss the formula $P = \frac{X - Y}{Y - Z}$ when $X > Y > Z$; when $X < Y < Z$; when $X = Y = Z$.

10. Discuss the formula of Ex. 9 when $X = Y$ and $Y > Z$; when $X > Y$ and $Y = Z$.

11. Discuss the formula of Ex. 9 when $X < Y$ and $Y > Z$; when $X = Y$ and $Y < Z$.

CHAPTER XV

GRAPHS

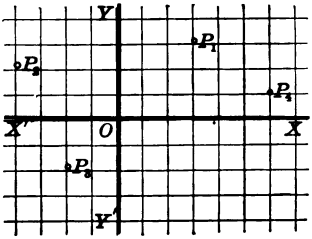
181. Location of Points on a Map. Points are located on a map by means of latitude and longitude. Latitude is stated in degrees north or south of the equator, longitude in degrees east or west of the prime meridian through Greenwich, England.

Thus to the nearest degree, the position of New York is 41° N. (that is, the latitude is 41° north of the equator) and 74° W. (that is, the longitude is 74° west of Greenwich). Similarly, the position of Chicago is 42° N. and 88° W.; of San Francisco, 38° N. and 122° W.; and of Paris, 49° N. and 2° E.

182. Location of Points on Paper. In a similar way we may locate points on paper. We may take two lines, one vertical and the other horizontal, and measure distances to the right and left of the one, and up and down from the other.

In this figure, P_1 is 3 units to the right of the vertical line YY' and 3 units above the horizontal line XX'; P_2 is 4 units to the left of YY' and 2 units above XX'; P_3 is 2 units to the left and 2 units below; and P_4 is 6 units to the right and 1 unit above.

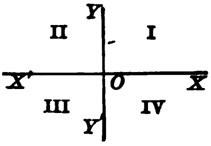
In the same way we may locate a chair on the floor with reference to the east and north wall, a spring in a field with reference to two fences meeting at right angles, or a point on the blackboard with reference to two lines perpendicular to each other.



Some of this work of locating points, given their distances from two lines intersecting at right angles, has already been done in connection with the study of the negative number (p. 33).

183. Axes. The two lines at right angles to each other, from which distances are measured in locating a point, are called *axes*.

The horizontal axis is called the *axis of x* and is lettered $X'X$, as in the figure at the right. The vertical axis is called the *axis of y* and is lettered YY'.



184. Origin. The intersection of the two axes is called the *origin*.

The origin is usually lettered O, as in this figure.

185. Quadrants. All that part of the plane above the axis of x and to the right of the axis of y is called the *first quadrant;* that above the axis of x and to the left of the axis of y, the *second quadrant;* that below the axis of x and to the left of the axis of y, the *third quadrant;* and that below the axis of x and to the right of the axis of y, the *fourth quadrant.*

186. Signs. Distances to the right of the axis of y or above the axis of x are considered *positive;* those to the left of the axis of y or below the axis of x are considered *negative.*

Thus we consider temperatures below zero as negative. Similarly, we might consider north latitude as positive and south latitude as negative, and east longitude as positive and west longitude as negative. Whatever direction is considered positive, the opposite direction is considered negative.

187. Designation of Points. In locating a point it is customary to state first the distance to the right or left of the vertical axis, parallel to the axis of x, and then the distance up or down from the horizontal axis, parallel to the axis of y.

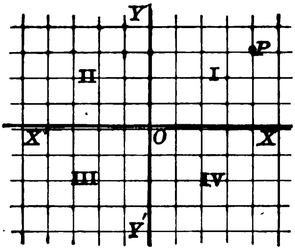
Thus the point 2, 3 is in the first quadrant, 2 units to the right of the axis of y and 3 units above the axis of x.

Similarly, -4, 5 is in the second quadrant, 4 units to the left of the axis of y and 5 units above the axis of x.

A point $-a$, $-b$ is evidently in the third quadrant, a units to the left of the axis of y and b units below the axis of x. A point a, $-b$ is in the fourth quadrant, a units to the right of the axis of y and b units below the axis of x.

188. Coördinates. The distances of a point to the right or left of the axis of y, and above or below the axis of x, are called the *coördinates* of the point.

Thus the point P is in the first quadrant. It is designated as the point (4, 3), and its coördinates are 4 and 3. A point (– 4, 3) would be represented in the second quadrant, and so on.



189. Abscissa. The distance of a point from the vertical axis, measured on or parallel to the axis of x, is called the *abscissa* of the point.

Thus the abscissa of P is 4.

The abscissa of a point in the second quadrant is negative. The abscissa of a point in the third quadrant is also negative. The abscissa of a point in the fourth quadrant is positive.

190. Ordinate. The distance of a point from the horizontal axis, measured on or parallel to the axis of y, is called the *ordinate* of the point.

The ordinate of P is 3.

The ordinate of a point in the first or second quadrant is positive. The ordinate of a point in the third or fourth quadrant is negative.

191. Plotting a Point. The representing of a point by means of its coördinates is called the *plotting* of the point.

To plot the point (– 2, – 3), take the abscissa – 2 and draw its ordinate – 3. The point is, therefore, in the third quadrant.

192. Coördinate Paper. Paper ruled in squares for convenience in plotting points is called *coördinate paper.*

Coördinate paper will be found of use in the representation of equations and in drawing many of the figures used in geometry.

Such paper can be purchased at stationers. It is also easily made by the student. Points may be plotted by the aid of a ruler without the use of coördinate paper.

Coördinate paper is also known as *cross-ruled paper*, *cross-section paper*, or *squared paper*.

Exercise 130. Plotting Points

Examples 1 to 6, oral — Examples 7 to 16, written

1. In what quadrant is the point (2, 5)?

2. In what quadrants are the points (– 2, – 6) and (4, – 2)?

3. Where is the point (0, 0)? (5, 0)? (0, 5)? (– 5, 0)?

4. What is the distance from (4, 0) to (– 4, 0)?

5. In what quadrant is (2, 7)? (– 2, 7)? (2, – 7)?

6. What is the distance from (4, 3) to (– 4, 3)? from (4, 3) to (4, – 3)? from (0, 0) to (0, 7)? from (– 2, 5) to (– 2, – 5)?

7. Plot the points (1, 3), (7, 6), (5, 2), (9, 1), (1, 9), (5, 5).

8. Plot the points (– 2, 3), (– 4, 6), (– 6, 4), (– 5, 2).

9. Plot the points (– 2, – 3), (– 4, – 6), (– 5, – 2), (– 6, – 9), (– 7, – 7).

10. Plot the points (2, – 3), (3, – 2), (4, – 6), (6, – 4), (7, – 7), (– 7, 9), (0, – 4), (1½, 2¼), (– 0.5, – 3).

11. Plot the points (4, 4), (2, 1), (4, 1), (2, 4), (– 4, – 4), (– 2, – 1), (– 4, – 1), (– 2, – 4).

12. Plot the points (3, 5), (2, 3), (1, 5), (2, 6), (5, 3), (2, 0), (– 3, – 5), (– 2, – 3), (– 1, – 5).

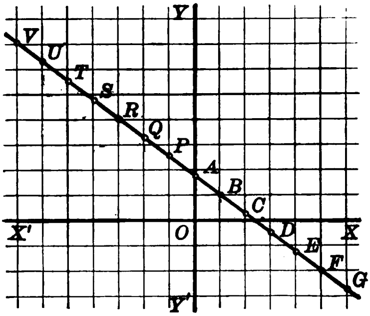
13. Plot the points (1, 4), (4, 5), (4, – 4), (1, 1), (1, – 1), (1, 3), (– 4, – 5), (– 1, – 1), (– 1, 4), (0, 4).

14. What letter is formed by joining (1, 1) and (1, 4), (1, 4) and (3, 1), (3, 1) and (5, 4), (5, 4) and (5, 1)?

15. Join in succession the points (0, 4), (1, 1), (2, 3), (3, 1), and (4, 4). What letter is formed?

16. Join (0, 0) and (3, 5), (3, 5) and (6, 0), (1.2, 2) and (4.8, 2), (7, 0) and (12, 5), (7, 5) and (12, 0), (13, 0) and (13, 5), (18, 4) and (17, 5), (17, 5) and (15, 5), (15, 5) and (14, 4), (14, 4) and (14, 3), (14, 3) and (18, 2), (18, 2) and (18, 1), (18, 1) and (17, 0), (17, 0) and (15, 0), (15, 0) and (14, 1). What word is spelled by these lines?

193. Plotting an Equation. Not only is it possible to plot a point, but it is also possible to plot an equation. For example, consider the equation $3x + 4y = 7$.



Solving for y we have $y = \frac{7 - 3x}{4}$. That is, y is a function of x (§ 46).

Evidently for any value that we may give to x we may find a corresponding value of y.

Thus if

$$x = 1, y = \frac{7 - 3 \times 1}{4} = 1.$$

We may now plot the point (1, 1) and we have one point that lies on the line. This is represented by *B* on the diagram.

Let $x =$	0	1	2	3	4	5	6
Then $y =$	$1\frac{3}{4}$	1	$\frac{1}{4}$	$-\frac{1}{2}$	$-1\frac{1}{4}$	-2	$-2\frac{3}{4}$
Point $=$	*A*	*B*	*C*	*D*	*E*	*F*	*G*

Let $x =$	-1	-2	-3	-4	-5	-6	-7
Then $y =$	$2\frac{1}{2}$	$3\frac{1}{4}$	4	$4\frac{3}{4}$	$5\frac{1}{2}$	$6\frac{1}{4}$	7
Point $=$	*P*	*Q*	*R*	*S*	*T*	*U*	*V*

We might also take fractional values of x and find corresponding values of y and corresponding points. All such points would be found to lie on a straight line, as shown in the diagram.

194. Equation of the First Degree. An equation of the first degree in two unknown quantities represents a straight line.

195. Graph. The line representing an equation is called the *graph* of the equation.

To *plot*, or *graph*, an equation means to draw the graph of the equation.

Exercise 131. Graphs of Equations

Examples 1 to 3, oral — Examples 4 to 22, written

1. In the equation $y = 4 - x$, find the value of y when $x = 1$; when $x = 2$; when $x = 4$; when $x = 7$; when $x = 0$; when $x = -4$.

2. In the equation $x = y - 5$, find the value of x when $y = 0$; when $y = 1$; when $y = 4$; when $y = 5$; when $y = 10$; when $y = -3$.

3. In the equation $x - y = 7$, what does x equal when $y = 7$? What does y equal when $x = 7$?

4. Make a table like that on page 257, giving values of y for fourteen values of x, in the equation $5x - 2y = 7$.

5. Make a similar table for the equation $3x - 4y = 9$.

6. Make a similar table for the equation $2x = 3y$.

7. Plot the equation given in Ex. 4.

8. Plot the equation given in Ex. 5.

9. Plot the equation given in Ex. 6.

Plot the following equations, fixing three points in each:

10. $x + 2y = 5$.

11. $x - 2y = 5$.

12. $3x + 4y = 7$.

13. $3x - 4y = 7$.

14. $5x + 7y = 12$.

15. $7x = 8y$.

16. $8x = 7y$.

17. $x = y$.

18. $2.7x + 1.2y = 3.9$.

19. $4.1x - 2.3y = 1.9$.

20. Plot the equation of which the graph cuts the axis of x at 5 and the axis of y at 3.

21. Plot the equation of which the graph passes through the points (0, 0) and (5, 5). At what angle does it seem to cut the axis of x?

22. Plot the equation of which the graph passes through the points (0, 0) and (−5, 5). At what angle does it seem to cut the axis of y?

196. Linear Equations. Because an equation of the first degree involving two unknown quantities has for its graph a straight line, all equations of the first degree are known as *linear equations.*

197. Variable. A quantity that, under the conditions of a problem, may take different values is called a *variable.*

In the equation $y = 4x - 2$ we may give to x various values and from these we may find corresponding values for y. This equation has therefore two variables, x and y.

198. Constant. A quantity that, under the conditions of a problem, has a fixed value is called a *constant.*

Variables are usually represented by the last letters of the alphabet, and constants by the first letters of the alphabet or by numerals.

199. Special Directions. Since two points exactly fix the position of a straight line, we need fix only two points in plotting a linear equation involving two variables. If we let $y = 0$ we have for the value of x the distance to the point where the graph cuts the axis of x. If we let $x = 0$ we have for the value of y the distance to the point where the graph cuts the axis of y.

In the equation

$$3x - 5y = 15,$$

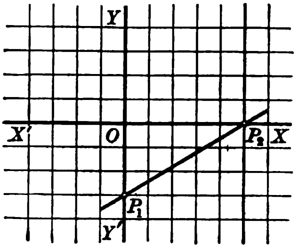
if $x = 0,\quad y = -3,$

and if $y = 0,\quad x = 5.$

Here $(0, -3)$ is the point P_1, and $(5, 0)$ is the point P_2.

Therefore the required graph is P_1P_2.

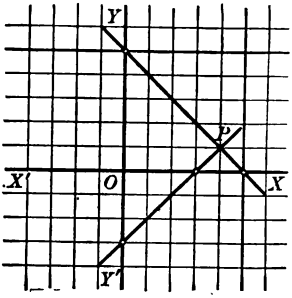
Any other two points may be taken to fix the line.



The equation $x = 5$ is equivalent to $x + 0 \cdot y = 5$, and hence, whatever value y has, x always equals 5. The graph is therefore parallel to the axis of y, 5 units to the right of the axis.

Similarly, the equation $x = -2$ represents a line 2 units to the left of the axis of y, and parallel to it; and $y = 4$ represents a line 4 units above the axis of x, and parallel to it.

200. Two Linear Equations. Two linear equations involving two variables are represented by two straight lines. These lines can intersect in only one point. Therefore, in general,



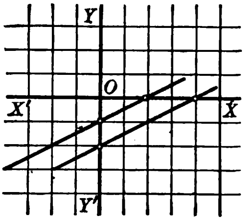
The graphs of two linear equations involving two variables have only one point in common;

Two linear equations involving two variables have only one pair of values of the variables in common.

Thus the graphs of $x + y = 5$ and $x - y = 3$ intersect at P. The coördinates of P are 4 and 1. Hence $x = 4$, $y = 1$.

There are an infinite number of points on each graph, but there is only one point on both graphs. Similarly, there are an infinite number of values of x and y that will satisfy each equation, but there is only one value that will satisfy both equations.

201. Inconsistent Equations. If we plot the equations $x - 2y = 4$ and $3x - 6y = 5$, we shall have two parallel lines. Such lines have no point in common. Considering the equations we see that the second one reduces to $x - 2y = 1\frac{2}{3}$. The equations are therefore *inconsistent,* since $x - 2y$ cannot equal both 4 and $1\frac{2}{3}$.



202. Equivalent Equations. If we plot the equations $2x + 4y = 5$, and $x + 2y = 2\frac{1}{2}$, we shall find that one graph coincides with the other. They have an infinite number of points in common, and therefore an infinite number of values of x and y satisfy the equations, every pair of roots of either being a pair of roots of the other. The equations are therefore *equivalent.*

Exercise 132. Graphs of Linear Equations

Examples 1 to 4, oral — Examples 5 to 36, written

1. What is the nature of the graph of $x = 2$? of $y = 2$?

2. What is the nature of the graph of $y = 4$? of $y = -4$?

3. State two points on the graph of $x + 4y = 8$.

4. State two points through which the graph of $x = y$ passes.

Plot the following equations, and solve by measuring the coördinates of the point of intersection of the graphs:

5. $x + 4y = 11$
$2x - y = 4$

6. $2x + 3y = 19$
$7x - 2y = 4$

7. $x + 5y = -3$
$2x - 3y = 20$

8. $2x - 9y = 23$
$5x + y = -13$

9. $x + 5y = 0$
$3x + 9y = -6$

10. $7x + 2y = 14$
$5x - 3y = -21$

11. $2x - 3y = 7$
$5x - 7y = 14$

12. $6x - 3y = 15$
$2x + 7y = 45$

13. Show by graphs that the equations $x + 4y = 6$ and $0.5x + 2y = 4$ are inconsistent.

14. Show by graphs that the equations $0.2x - 0.5y = 6$ and $x - 2.5y = 30$ are indeterminate.

15. Show by graphs that the three equations $x + y = 6$, $2x - y = 0$, and $5x + 3y = 22$ have a common root. Find the root.

16. Show by graphs that the three equations $x + y = 5$, $2x - 3y = 20$, and $3x + y = 2$ have no root common to all three.

17. If $x + 5y = 2.1$, y equals what function of x? x equals what function of y? If $2x - y = 2$, y equals what function of x? x equals what function of y?

18. Taking $y = f(x)$ and $y = F(x)$ as found in Ex. 17, plot these two equations, and solve by measuring the coördinates.

Plot the following equations, and determine what sets admit of solution and what do not:

19. $3x + 6y = 7$
$2x - 4y = 5$

20. $3x + 6y = 7$
$2x + 4y = 5$

21. $3x + 6y = 7$
$2x + 4y = 4\frac{2}{3}$

22. $3x - 6y = 7$
$2x + 4y = 5$

23. $3x - 6y = 7$
$2x - 4y = 5$

24. $2x + 7y = 27$
$5x - 2y = 9$

25. $x + y = 9$
$x - y = 1$

26. $x + 7y = 14$
$7x - y = 48$

27. $3x + 8y = 12$
$4.5x + 12y = 16$

28. $5x - 7y = 6$
$6x - 8.4y = 8$

29. $2x - 5y = 3$
$5x - 2y = 3$

30. $5x + 12.8y = 35$
$3\frac{1}{7}x - 4.8y = 22$

Determine by means of graphs whether one or more of the following sets of equations have a root common to all three equations of the set:

31. $x + 2y = 8$
$y - x = 1$
$4x + y = 6$

32. $5x - y = 7$
$x + 7y = 23$
$2x + y = 5$

33. $4x + y = 21$
$4y + x = 9$
$4x + 4y = 12$

34. $x + y = 9$
$x - y = 5$
$2x + 3y = 20$

35. When we consider the graphs of two linear equations containing only x and y, how does it appear that such equations are usually simultaneous? Draw the graphs of two equations to illustrate this fact.

36. When we consider the graphs of three linear equations containing only x and y, how does it appear that such equations are usually not simultaneous? Draw the graphs of three equations to illustrate this fact.

CHAPTER XVI

POWERS AND ROOTS

203. Power. The product of several equal factors is called a *power* of the factor.

204. Laws of Exponents. The laws of exponents are these:

1. *Products:* $a^m a^n = a^{m+n}$. § 63

2. *Quotients:* $a^m \div a^n = a^{m-n}$. § 72

3. *Powers of products:* $(ab)^m = a^m b^m$.

For
$$\begin{aligned}(ab)^m &= ab \cdot ab \cdot ab \cdots m \text{ times}\\ &= (aaa \cdots \text{to } m \text{ factors})(bbb \cdots \text{to } m \text{ factors})\\ &= a^m b^m.\end{aligned}$$

Conversely, $a^m b^m = (ab)^m$.

Similarly, $(abc)^m = a^m b^m c^m$, and so on for any number of factors.

4. *Powers of quotients:* $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.

For
$$\begin{aligned}\left(\frac{a}{b}\right)^m &= \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} \cdots \text{to } m \text{ factors}\\ &= \frac{aaa \cdots \text{to } m \text{ factors}}{bbb \cdots \text{to } m \text{ factors}}\\ &= \frac{a^m}{b^m}.\end{aligned}$$

Conversely, $\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$.

5. *Powers of powers:* $(a^m)^n = a^{mn}$.

For
$$\begin{aligned}(a^m)^n &= (aaa \cdots \text{to } m \text{ factors})^n\\ &= a^n a^n a^n \cdots \text{to } m \text{ factors, by Law 3}\\ &= a^{n+n+n \cdots \text{to } m \text{ terms}}, \text{ by Law 1}\\ &= a^{mn}.\end{aligned}$$

205. Law of Signs. From the law of signs in multiplication (§ 64) we have the following law:

Even powers of a positive or a negative number are positive; odd powers have the same sign as the number itself.

Thus $(+3)^2 = +9$, $(+a)^2 = +a^2$, $(\pm a)^4 = +a^4$,
and $(-3)^2 = +9$, $(-a)^2 = +a^2$, $(\pm a)^{10} = +a^{10}$.
Likewise, $(+2)^3 = +8$, $(+a)^3 = +a^3$, $(+a)^5 = +a^5$,
and $(-2)^3 = -8$, $(-a)^3 = -a^3$, $(-a)^5 = -a^5$.

Exercise 133. Powers of Monomials

Examples 1 to 6, oral — Examples 7 to 35, written

1. State the product of $a^2 \cdot a^3$; of $a^4 \cdot a^7$; of $x^2 \cdot x^3 \cdot x^4$.

2. State the product of $a^p \cdot a^q$; of $a^m \cdot a$; of $a^p \cdot a^q \cdot a^r$.

3. State the quotient of $a^5 \div a^2$; of $3^3 \div 3^2$; of $a^p \div a^q$.

4. Express without the parentheses: $(ab)^p$; $(abcd)^m$; $(xyz)^3$.

5. Express without the parentheses: $(a^p)^q$; $(a^q)^p$; $(a^{mn})^p$; $(a^m b^n)^p$; $(a^x b^y c^z)^2$; $(a^w b^x c^y d^z)^4$.

6. Express without the parentheses: $\left(\frac{2}{3}\right)^2$; $\left(\frac{a}{b}\right)^2$; $\left(\frac{a^m}{b^m}\right)^p$; $\left(\frac{a^m}{b^m}\right)^3$.

Express the following without the parentheses:

7. $(a^2)^5$.
8. $(a^5)^2$.
9. $(2^3)^2$.
10. $(3^2)^3$.
11. $(a^2x^3)^7$.
12. $(a^5b^9)^{13}$.
13. $(-a^2)^4$.
14. $(-2^3)^2$.
15. $(-ab)^9$.
16. $(-a^2b^3)^3$.
17. $(-a^m b^n)^2$.
18. $(-a^m b^n)^3$.
19. $(a^m b^n)^p$.
20. $(a^5b^7)^m$.
21. $(-a^9b^9)^9$.
22. $(-a^3b^3)^3$.
23. $(-a^m b^n)^{15}$.
24. $(-a^m b^n)^{16}$.
25. $(ab^2c^3d^4)^5$.
26. $(4a^2b^3c)^3$.
27. $(-2a^5b^5)^4$.
28. $(-3a^7b^9)^3$.
29. $(-2a^9b^3)^6$.
30. $(-2x^7y^9)^5$.

31. Is $2n$ even or odd? Is $2n+1$ even or odd? What is the sign of $(-a)^{2n}$? of $(-a)^{2n+1}$?

Express the following without the parentheses:

32. $\left(\frac{a^7}{b^9}\right)^6$.
33. $\left(-\frac{p^3}{q^{10}}\right)^5$.
34. $\left(-9\frac{x^5}{y^7}\right)^2$.
35. $\left(-\frac{a^m}{b^m}\right)^{2n+1}$.

206. Binomial Theorem. By actual multiplication we have:

$$(a+b)^2 = a^2 + 2\,ab + b^2,$$
$$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3,$$
$$(a+b)^4 = a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4,$$
$$(a+b)^5 = a^5 + 5\,a^4b + 10\,a^3b^2 + 10\,a^2b^3 + 5\,ab^4 + b^5.$$

In these results it is seen that:

1. *The number of terms is greater by one than the exponent of the power to which the binomial is raised.*

2. *In the first term the exponent of a is the same as the exponent of the power to which the binomial is raised, and decreases by one in each succeeding term.*

3. *The letter b appears in the second term, with an exponent 1, and the exponent increases by one in each succeeding term.*

4. *The coefficient of the first term is 1, and the coefficient of the second term is the same as the exponent of the binomial.*

5. *The coefficient of each term after the second is found from the preceding term by multiplying the coefficient of that term by the exponent of a and dividing the product by a number greater by one than the exponent of b.*

The above law is called the *binomial theorem.*

If b is negative, the terms in which the *odd* powers of b occur are *negative.* Thus

$$(a-b)^2 = a^2 - 2\,ab + b^2;$$
$$(a-b)^3 = a^3 - 3\,a^2b + 3\,ab^2 - b^3;$$
$$(a-b)^6 = a^6 - 6\,a^5b + 15\,a^4b^2 - 20\,a^3b^3 + 15\,a^2b^4 - 6\,ab^5 + b^6.$$

In these cases it will be noticed that the terms are alternately positive and negative.

The most important part of the Binomial Theorem is 5, and this should be mastered. The theorem is true for any positive integral exponent. Other exponents are considered later in the student's work in mathematics.

A complete proof of the Binomial Theorem is not practicable at this point in our work. Enough has been shown, however, to convince us of the probable truth of the statements above made. A proof of the theorem is given on page 378.

1. Raise $2x - 3$ to the second power.

Since $$(a - b)^2 = a^2 - 2ab + b^2,$$

therefore $$(2x - 3)^2 = (2x)^2 - 2 \cdot (2x) \cdot 3 + 3^2$$
$$= 4x^2 - 12x + 9.$$

Check. Let $x = 1$.

Then $$(-1)^2 = 1 = 4 - 12 + 9.$$

2. Raise $\frac{3}{5}x^2y^3 - \frac{2}{3}$ to the third power.

Since $$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3,$$

therefore
$$(\tfrac{3}{5}x^2y^3 - \tfrac{2}{3})^3 = (\tfrac{3}{5}x^2y^3)^3 - 3 \cdot (\tfrac{3}{5}x^2y^3)^2 \cdot \tfrac{2}{3} + 3 \cdot (\tfrac{3}{5}x^2y^3) \cdot (\tfrac{2}{3})^2 - (\tfrac{2}{3})^3$$
$$= \tfrac{27}{125}x^6y^9 - \tfrac{18}{25}x^4y^6 + \tfrac{4}{5}x^2y^3 - \tfrac{8}{27}.$$

Check. Let $x = 1$, $y = 1$.

Then $$(\tfrac{3}{5} - \tfrac{2}{3})^3 = (-\tfrac{1}{15})^3 = -\tfrac{1}{3375},$$

and $$\tfrac{27}{125} - \tfrac{18}{25} + \tfrac{4}{5} - \tfrac{8}{27} = \tfrac{37}{125} - \tfrac{8}{27} = -\tfrac{1}{3375}.$$

3. Raise $2x - y^2$ to the fifth power.

Since $$(a - b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5,$$

therefore $$(2x - y^2)^5 = (2x)^5 - 5(2x)^4y^2 + 10(2x)^3y^4 - 10(2x)^2y^6$$
$$+ 5(2x)y^8 - y^{10}$$
$$= 32x^5 - 80x^4y^2 + 80x^3y^4 - 40x^2y^6 + 10xy^8 - y^{10}.$$

Check. Let $x = 1$, $y = 1$.

Then both members of the equation reduce to 1.

Exercise 134. Powers of a Binomial

Examples 1 to 8, oral — Examples 9 to 24, written

1. $(a + 1)^2$.
2. $(a - 2)^2$.
3. $(a + 3)^2$.
4. $(a^2 - b^2)^2$.
5. $(a - 1)^3$.
6. $(a^2 - 1)^3$.
7. $(a^3 + 1)^2$.
8. $(a^3 - 1)^2$.
9. $(4a + 5b)^2$.
10. $(5a^2 - 3b)^2$.
11. $(3a^4 + 2)^2$.
12. $(\frac{2}{3}x - 3y)^3$.
13. $(\frac{3}{4}x^2 + \frac{2}{3}y)^3$.
14. $(\frac{3}{5}x^3 - \frac{1}{2}y^2)^3$.
15. $(0.1x^2 + 0.2y^2)^3$.
16. $(2.1x^2 - 0.3y^2)^3$.
17. $(2a + 1)^4$.
18. $(2m^2 - 3n)^4$.
19. $(\frac{1}{2}p + \frac{2}{3}q)^5$.
20. $(\frac{3}{4}x^2 - \frac{2}{3}y^2)^5$.
21. $(a - 3b)^6$.
22. $(x + 3y^2)^6$.
23. $(2x + y^3)^7$.
24. $(1.1x + 2.1y)^4$.

207. Root. Any one of the equal factors whose product is a given number is called a *root* of the number.

The term is already familiar from arithmetic and has therefore been used whenever necessary in the previous work (§ 7).

The root is called a square root, cube root, fourth root, or rth root, according as it is one of two, three, four, or r equal factors.

The word *root* is used in two different senses in algebra. The other use is to represent the value of the unknown quantity in an equation. Thus the square root of 25 is 5, and the root of the equation $x - 7 = 23$ is 30.

208. Radical Sign. The common symbol for root $(\sqrt{\;})$ is called the *radical sign.*

Thus, $\sqrt[4]{625} = 5$, because $625 = 5 \cdot 5 \cdot 5 \cdot 5$. In this case 4 is called the *index* of the root. In $\sqrt[n]{a - b}$, n is the index of the root. In the case of a square root, such as $\sqrt{4}$, the index is not written.

209. Imaginary Number. Since any even power of a positive or a negative number is positive (§ 205), therefore an even root of a negative number cannot be positive and cannot be negative.

We can merely indicate such roots thus: $\sqrt{-1}$, $\sqrt{-3}$, $\sqrt[4]{-2}$. Such numbers are treated later in our work.

An indicated square root of a negative number is called an *imaginary number.*

210. Real Number. A number that does not contain an imaginary number is called a *real number.*

A real number may be an integer (whole number) like 3 or -7, a fraction like $\frac{2}{3}$, $\frac{5}{2}$, $-\frac{5}{8}$, or 0.7, a mixed number like $1\frac{1}{2}$ or -2.75, or such a number as $\sqrt{3}$ or π; but $2 + 3\sqrt{-5}$ is not a real number.

211. Rational Number. An integer or the quotient of two integers is called a *rational number.*

For example, 3, -7, $\frac{2}{3}$, $\frac{5}{2}$, 1.25, and -0.75 are rational numbers.

212. Irrational Number. A real number that is not rational is called an *irrational number.*

For example, $\sqrt{2}$ and $\sqrt{\frac{1}{2}}$ are both irrational. Similarly, numbers like $-2\sqrt{3}$ and $\frac{2}{3}\sqrt[3]{\frac{1}{2}}$ are irrational.

213. Signs of Roots. Since $(+2)\cdot(+2)=4$, and $(-2)(-2)=4$, we see that 4 has two square roots, $+2$ and -2. For the same reason it is evident that

Every positive number has two real square roots which have the same absolute values but have opposite signs.

Similarly, every positive number has at least two real fourth roots, two real sixth roots, and so on for any roots indicated by an even number.

For example, 9 has two real square roots, $+3$ and -3; and 16 has two fourth roots, $+2$ and -2, since $(+2)^4=16$, and $(-2)^4=16$. Imaginary roots are discussed later.

As a matter of fact every number has three cube roots, four fourth roots, five fifth roots, and so on. Some of these roots are imaginary, but we do not need to consider such roots at the present time.

Furthermore, since $(+2)^3=+8$, and $(-2)^3=-8$, we see that $\sqrt[3]{+8}=+2$, and $\sqrt[3]{-8}=-2$. For the same reason it is evident that

An odd root of a number has the same sign as the number itself.

For example, because $(-2)^5=-32$, therefore $\sqrt[5]{-32}=-2$, while $\sqrt[5]{+32}=+2$.

214. Principal Root. The one real root of a number, if it has but one, or its real positive root, if it has two real roots, is called the *principal root* of the number.

Thus the principal square root of 4 is $+2$, although there is another root, -2; and the principal cube root of -27 is -3, although it will later be shown that there are two imaginary roots.

215. Radical Sign and Principal Root. It is to be understood in algebra that *the radical sign indicates the principal root.*

Thus $\sqrt{4}=2$, and it is incorrect to write $\sqrt{4}=-2$, although 4 has two square roots. If we wish to indicate the negative root, we write $-\sqrt{4}=-2$, and if we wish to indicate both roots, we write $\pm\sqrt{4}=\pm 2$.

Therefore if we have to simplify $\sqrt{4}+\sqrt{9}+\sqrt{16}$, we have $2+3+4=9$, and not $\pm 2\pm 3\pm 4$, which has several possible values.

Similarly, if we have the equation $x^2=5$ we should write $x=\pm\sqrt{5}$ and not merely $x=\sqrt{5}$.

216. Square Root of a Polynomial. If we square $a + b$ the result is, as shown in § 91, $a^2 + 2ab + b^2$. In extracting the square root of $a^2 + 2ab + b^2$ we reverse the process of squaring, thus:

$$\begin{array}{lll|l}
\text{The given square} = & & a^2 + 2ab + b^2 & \underline{a + b} = \text{Root} \\
\text{The square of } a = & & \underline{a^2} & \\
\text{Trial divisor} = & 2a & \multicolumn{1}{|l}{2ab + b^2 = \text{First remainder}} & \\
\text{Complete divisor} = & 2a + b & \multicolumn{1}{|l}{\underline{2ab + b^2} = b(2a + b)} &
\end{array}$$

The first term of the root is evidently a, because the square of a is a^2.

Since in squaring a binomial we have the square of the first term plus twice the product of the first and second terms, etc., we have in $2ab$ twice the product of a and the second term. We therefore divide by $2a$ to find the second term. Since $2ab \div 2a$ equals b, b is the second term.

In squaring $a + b$ we have $a^2 + b(2a + b)$. We therefore add b to $2a$, and multiply $2a + b$ by b, thus completing the square of $a + b$.

The names *trial divisor* and *complete divisor* are used as above shown, being convenient names brought into algebra from arithmetic.

1. Find the square root of $9x^2 - 30xy^2 + 25y^4$.

$$\begin{array}{lll|l}
\text{The given square} = & & 9x^2 - 30xy^2 + 25y^4 & \underline{3x - 5y^2} = \text{Root} \\
\text{The square of } 3x = & & \underline{9x^2} & \\
\text{Trial divisor} = & 6x & \multicolumn{1}{|l}{-30xy^2 + 25y^4} & \\
\text{Complete divisor} = & 6x - 5y^2 & \multicolumn{1}{|l}{\underline{-30xy^2 + 25y^4}} &
\end{array}$$

The student should compare this, step by step, with the example explained above, answering the following questions:

How is the first term of the root found?

Why is $6x$ taken as the divisor?

How is the second term found?

Why is $-5y^2$ added to $6x$?

Why is $6x - 5y^2$ multiplied by $-5y^2$?

2. Find the square root of $81x^2y^2 - 90xy + 25$.

$$\begin{array}{ll|l}
 & 81x^2y^2 - 90xy + 25 & \underline{9xy - 5} \\
 & \underline{81x^2y^2} & \\
18xy & \multicolumn{1}{|l}{-90xy + 25} & \\
18xy - 5 & \multicolumn{1}{|l}{\underline{-90xy + 25}} &
\end{array}$$

This example is merely illustrative. In such a simple case we would ordinarily find the square root by inspection.

3. Find the square root of $4x^4 - 12x^2y + 9y^2 + 4x^2z^3 - 6yz^3 + z^6$.

$$\begin{array}{l|l|l}
 & 4x^4 - 12x^2y + 9y^2 + 4x^2z^3 - 6yz^3 + z^6 & \underline{2x^2 - 3y + z^3} \\
 & 4x^4 & \\ \hline
4x^2 & -12x^2y + 9y^2 & \\
4x^2 - 3y & -12x^2y + 9y^2 & \\ \hline
4x^2 - 6y & & 4x^2z^3 - 6yz^3 + z^6 \\
4x^2 - 6y + z^3 & & 4x^2z^3 - 6yz^3 + z^6 \\ \hline
\end{array}$$

The first term of the root is $2x^2$.

Subtracting the square of $2x^2$, which is $4x^4$, the remainder contains twice the first term times the second, plus the square of the second. Dividing by $4x^2$, which is twice $2x^2$, the second term is $-3y$.

Proceeding as before and subtracting, we may consider the second remainder as twice the product of $(2x^2 - 3y)$ times the next term, considering the binomial $2x^2 - 3y$ as the first term.

Dividing by twice $2x^2 - 3y$, or by $4x^2 - 6y$, we find the next term to be z^3.

Proceeding as before and subtracting, there is no remainder. Hence the square root is $2x^2 - 3y + z^3$.

We may check this result by squaring it, or we may let $x = 2$, $y = 1$, $z = 1$ (or any other convenient values). Then we have $6^2 = 36$.

If the terms of the given polynomial are not arranged according to the ascending or descending powers of some letter, make this arrangement before beginning the work.

217. Extracting the Square Root. Therefore, in extracting the square root of a polynomial,

Arrange the terms according to the powers of some letter.

Find the principal square root of the first term, and subtract its square from the polynomial.

Divide the first term of the remainder by twice the first term of the root, and write the quotient as the second term of the root.

Multiply the sum of twice the first term and the second term of the root by the second term, thus completing the square of the sum of the first two terms, and subtract this product.

Consider what has now been found as the first part of the root, and proceed as before.

Exercise 135. Square Root of a Polynomial

Examples 1 to 8, oral — Examples 9 to 20, written

1. What is the square root of $4a^2 - 4a + 1$?

2. What is the square root of $9a^4 + 6a^2 + 1$?

3. What are the two square roots of $a^2b^2c^2 - 4abc + 4$?

For both square roots, inclose the root in parentheses preceded by the ± sign.

Find the square roots of the following:

4. $x^2 - 6x + 9$.

5. $a^2 - 4ab + 4b^2$.

6. $16m^2 + 24mn + 9n^2$.

7. $x^4 + 16x^2y^2 + 64y^4$.

8. It is known that $x^4 + 2x^3 + 3x^2 + 2x + 1$ is the square of a trinomial. Looking at the first term, state the first term of the square root of the polynomial. Looking at the last term of the polynomial, state the last term of the root. Dividing the second term of the polynomial by twice the first term of the root, state the second term of the root. What is the square root of the polynomial? Check the result by written work.

Find the square roots of the following:

9. $a^4 - 2a^3 + 3a^2 - 2a + 1$.

10. $x^4 + 2x^3 + 5x^2 + 4x + 4$.

11. $a^4 - 2a^3 + 5a^2 - 4a + 4$.

12. $x^2 + 2xy + y^2 + 2x + 2y + 1$.

13. $x^2 + 2xy + y^2 + 2xz + 2yz + z^2$.

14. $x^4 - 6x^3 + 17x^2 - 24x + 16$.

15. $9x^4 - 12x^3 + 34x^2 - 20x + 25$.

16. $4x^6 - 12x^5 - 7x^4 + 44x^3 - 14x^2 - 40x + 25$.

17. $25x^6 - 30x^5y + 9x^4y^2 + 10x^3y^2 - 6x^2y^3 + y^4$.

18. $29a^4b^2 + 16b^6 - 12a^5b - 46a^3b^3 + 4a^6 - 40ab^5 + 49a^2b^4$.

19. $16x^2 + 8xy + y^2 + 8xz + 2yz + z^2 + 8x + 2y + 2z + 1$.

20. $x^2 + 8xy + 16y^2 + 2xz + 8yz + z^2 + 2x + 8y + 2z + 1$.

218. Square Root of a Polynomial containing Fractions. If a polynomial contains powers and reciprocals of the same letter, the order of arrangement in descending powers of the letter is as follows:

$$\cdots x^4,\ x^3,\ x^2,\ x,\ 1,\ \frac{1}{x},\ \frac{1}{x^2},\ \frac{1}{x^3},\ \frac{1}{x^4},\ \cdots$$

Find the square root of $x^2 + 13 + 6x + \frac{4}{x^2} + \frac{12}{x}$.

Arrange according to the descending powers of x, and proceed as before.

$$\begin{array}{l|l|l}
 & x^2 + 6x + 13 + \frac{12}{x} + \frac{4}{x^2} & x + 3 + \frac{2}{x} \\
 & x^2 & \\
\hline
2x & 6x + 13 & \\
2x + 3 & 6x + 9 & \\
\hline
2x + 6 & 4 + \frac{12}{x} + \frac{4}{x^2} & \\
2x + 6 + \frac{2}{x} & 4 + \frac{12}{x} + \frac{4}{x^2} & \\
\hline
\end{array}$$

Exercise 136. Square Root of a Polynomial

Examples 1 to 6, oral — Examples 7 to 28, written

1. Find the square root of $x^2 + 2 + \frac{1}{x^2}$; of $4x^2 + 4 + \frac{1}{x^2}$.

2. What are the two square roots of $x^2 - 6 + \frac{9}{x^2}$?

Find the square roots of the following:

3. $\frac{x^2}{9} + \frac{2x}{3} + 1$.

4. $1 + \frac{6}{x} + \frac{9}{x^2}$.

5. $x^2 + 2 \cdot \frac{x}{y} + \frac{1}{y^2}$.

6. $\frac{1}{x^2} - \frac{2}{xy} + \frac{1}{y^2}$.

7. $81x^2 - 126 + \frac{49}{x^2}$.

8. $4a^4 + 20a^2 - 3 - \frac{70}{a^2} + \frac{49}{a^4}$.

9. $4a^4 - 12a + 4 + \frac{9}{a^2} - \frac{6}{a^3} + \frac{1}{a^4}$.

10. $\frac{a^4}{4} - \frac{a^2b^3}{3} + \frac{b^6}{9} + \frac{a^2c^4}{4} - \frac{b^3c^4}{6} + \frac{c^8}{16}$.

11. $13x^4 + 13x^2 + 4x^6 - 14x^3 + 4 - 4x - 12x^5$.

Find the square roots of the following:

12. $0.04\,a^4 + 0.12\,a^3 + 0.29\,a^2 + 0.3\,a + 0.25$.

Since $0.2 \times 0.2 = 0.04$, therefore $\sqrt{0.04} = 0.2$.

13. $0.01a^6 + 0.04a^5 + 0.1\,a^4 + 0.2\,a^3 + 0.25\,a^2 + 0.24\,a + 0.16$

14. $x^2 + \frac{2\,x}{y} + \frac{2\,x}{z} + \frac{2}{yz} + \frac{1}{y^2} + \frac{1}{z^2}$.

15. $9\,a^2 - \frac{12\,a}{b} + \frac{4}{b^2} + \frac{6\,a}{c} - \frac{4}{bc} + \frac{1}{c^2}$.

16. $25\,a^4 + \frac{9}{b^2} + \frac{4}{c^4} + \frac{30\,a^2}{b} - \frac{20\,a^2}{c^2} - \frac{12}{bc^2}$.

17. $\frac{x^2}{4\,z^4} + \frac{4\,z^2}{25\,x^4} + \frac{2}{5\,xz} - \frac{12\,z}{5\,x^2} - \frac{3\,x}{z^2} + 9$.

18. $\frac{x^2}{y^2} - \frac{4\,xz}{ay} + \frac{4\,z^2}{a^2} + \frac{6\,qx}{by} + \frac{9\,q^2}{b^2} - \frac{12\,qz}{ab}$.

19. $x^6y^6 - \frac{2\,x^3y^{12}}{3\,a^3} + \frac{y^{18}}{9\,a^6} + \frac{4\,x^3y^{18}}{a^9} - \frac{4\,y^{24}}{3\,a^{12}} + \frac{4\,y^{30}}{a^{18}}$.

20. $\frac{9\,x^2y^2}{49\,a^2b^2} - \frac{15\,xy}{28\,ab} + \frac{203}{192} - \frac{35\,ab}{36\,xy} + \frac{49\,a^2b^2}{81x^2y^2}$.

21. $\frac{1}{4}\,p^4 - \frac{1}{3}\,p^2q^3 + \frac{1}{9}\,q^6 + \frac{1}{4}p^2r^4 + \frac{1}{16}\,r^8 - \frac{1}{6}\,q^3r^4$.

22. $\frac{1}{4}\,m^4 + \frac{1}{3}\,m^3n - \frac{5}{36}\,m^2n^2 - \frac{1}{6}\,mn^3 + \frac{1}{16}\,n^4$.

23. $4\,a^6 - 14\,a^2b^4 + 25\,b^6 - 7\,a^4b^2 + 12\,a^5b + 40\,ab^5 - 44\,a^3b^3$.

24. $4p^6 - 20p^5q - 3\,p^4q^2 + 9\,q^6 - 42\,pq^5 + 19\,p^2q^4 + 82\,p^3q^3$.

25. $a^2 + 2\,ab + b^2 + 2\,ac + 2\,bc + c^2 + 2\,ad + 2\,bd + 2\,cd + d^2$.

26. If the square root of $a^4 + 8\,a^3 + 10\,a^2 - 24\,a + 9$ is $a^2 + xa - y$, what are the values of x and y?

27. If the square root of $a^2 + xa + 9$ is $a + y$, what are the values of x and y?

28. It is known that $x^4 + 6\,x^3 + 19\,x^2 + 30\,x + 25$ is an exact square of a trinomial. Without going through the complete process of extracting the square root, state the first term of the root; the last term of the root; the second term of the root; the entire root. Check the result.

219. Arithmetical Square Root. The first step in extracting the square root of a number is to separate the figures of the number into groups of two figures each, called *periods*.

Since $1 = 1^2$, $100 = 10^2$, $10{,}000 = 100^2$, and so on, it is evident that the square root of any number between 1 and 100 lies between 1 and 10; of any number between 100 and 10,000 lies between 10 and 100. In other words, the square root of any integral number expressed by *one* or *two* figures is a number of *one* figure; expressed by *three* or *four* figures is a number of *two* figures; and so on.

Therefore if an integral number is separated into periods of two figures each, from the right to the left, the number of figures in the square root will be equal to the number of the periods of figures. The last period at the left may have two figures or one figure; for example, 22 09, and 7 89 04 81.

1. Find the square root of 2209.

We first recall that if $t =$ tens and $u =$ units, we have $(t + u)^2 = t^2 + 2tu + u^2$.

Separating into periods, we see that there will be two integral places in the root.

The first period, 22, contains t^2. And since the greatest square in 22 is 16, therefore $\sqrt{16}$, or 4, is the tens' figure of the root, or t.

$$\begin{array}{rl} & 22\ 09(47 \\ & 16 \\ \hline 80 & 6\ 09 \\ 87 & 6\ 09 \\ \hline \end{array}$$

Subtracting t^2, the remainder contains $2tu + u^2$. Therefore if we divide by $2t$ (that is, by 80, which is 2×4 tens), we shall find approximately u, the units. Dividing 609 by 80 (or 60 by 8), we have 7 as the units' figure.

Since $2tu + u^2 = (2t + u)u$, that is, $2 \times 40 \times 7 + 7^2 = (2 \times 40 + 7) \times 7$, we add 7 to 80 and multiply the sum by 7. The product is 609, thus completing the square of 47.

2. Find the square root of 7,890,481.

$$\begin{array}{rl} & 7\ 89\ 04\ 81(2809 \\ & 4 \\ \hline 40 & 3\ 89 \\ 48 & 3\ 84 \\ \hline 5600 & 5\ 04\ 81 \\ 5609 & 5\ 04\ 81 \\ \hline \end{array}$$

When the third period, 04, is brought down, and the divisor, 560, formed, the next figure of the root is 0, because 560 is not contained in 504. Therefore 0 is placed in the root, another 0 is annexed to the divisor, and the next two figures, 81, are brought down.

The work then proceeds as before, the trial divisor now being 5600, which is twice 280 tens. The entire root is thus found to be 2809.

220. Square Roots of Decimals. If the square root of a number has decimal places, the number itself will have twice as many.

Thus if 0.25 is the square root of some number, the number will be 0.25^2, or 0.25 × 0.25, or 0.0625. Hence if a given number contains a decimal, we separate the number into periods of two figures each, beginning at the decimal point and proceeding toward the left for the integral part, and toward the right for the decimal. The last period of the decimal must have two figures, a zero being annexed if necessary.

```
           52.27 29 (7.23
           49
     140    3 27
     142    2 84
     1440     43 29
     1443     43 29
```

Find the square root of 52.2729.

We see at once that the root will have one integral place. Furthermore, if it is a perfect square it will have two decimal places, since the square of thousandths is millionths.

Exercise 137. Arithmetical Square Roots

Examples 1 to 3, oral — Examples 4 to 26, written

1. State the squares of 11, 12, 13, 15, 20, 21, 25, 50.
2. State the square roots of 144, 169, 400, 900, 1600, 3600.
3. What is the square root of 121? of 1.21? of 0.0121?

Extract the square root of:

4. 190,969.
5. 743,044.
6. 401,956.
7. 758,641.
8. 117,649.
9. 804,609.
10. 194,481.
11. 173,056.
12. 174,724.
13. 509,796.
14. 1036.84.
15. 82.2649.
16. 0.063001.
17. 1.5129.
18. 2.6244.
19. 3.9204.
20. 462.25.
21. 0.003969.
22. 0.182329.
23. 0.054756.

Find the side of a square whose area is:

24. 12,321 sq. ft.
25. 8046.09 sq. in.

26. What is the perimeter of a square whose area is 1944.81 sq. in.?

221. Approximate Square Roots. If a number is not a perfect square, zeros may be annexed and an approximate value of the root found.

For example, extract to three places of decimals the square root of 19.

	19.00 00 00 (4.3588 +
	16
80	3 00
83	2 49
860	51 00
865	43 25
8700	7 75 00
8708	6 96 64
87160	78 3600

In this example we proceed in the usual way, annexing pairs of zeros for each decimal place in the root. We carry the work to four decimal places, so as to find the nearest approximation for three places. The result is found to be 4.359 – ; that is, it is nearer 4.359 than 4.358.

222. Summary of Square Root. We may summarize the process of square root as follows:

Separate the number into periods of two figures each, beginning at the decimal point.

Find the greatest square in the period at the left and write its root for the first figure of the required root.

Square this root, subtract the result from the period at the left, and to the remainder annex the next period for a dividend.

For a partial divisor double the root already found, considered as tens, and divide the dividend by it. The quotient (or the quotient diminished slightly if necessary) will be the next figure of the root.

To this partial divisor add this next figure of the root for a complete divisor. Multiply the complete divisor by this next figure of the root, subtract the product from the dividend, and to the remainder annex the next period for a new dividend.

Proceed in this manner until all the periods have been thus annexed. The result will be the square root required.

[illegible] the square [illegible] square roots [illegible]

[illegible] ter to multiply [illegible] ke it such, or [illegible] are root. For [illegible] requires us to

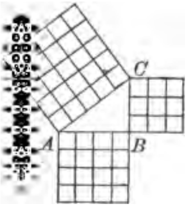
$$[illegible] \frac{\sqrt{2}}{2} = \frac{1}{2}\sqrt{2} = \frac{1}{2} [illegible]$$

[illegible] f 0.5.

[illegible] right angle [illegible] *right-angled* [illegible]

[illegible] work in arith- [illegible] relation to the [illegible]

[illegible] opposite the [illegible]



[illegible] ght triangles.

[illegible] he hypotenuse [illegible] es.

[illegible] sum of the [illegible]

[illegible] re root of the [illegible] other side.

[illegible] etry. It is said [illegible] e Greek mathe- [illegible] a.

Exercise 138. Arithmetical Square Roots

Examples 1 and 2, oral — Examples 3 to 39, written

1. Find the square root of $\frac{25}{144}$, $\frac{36}{121}$, $\frac{49}{81}$, $\frac{64}{121}$.

2. Find the square root of $\frac{121}{169}$, $\frac{441}{625}$, $\frac{100}{121}$, $\frac{121}{400}$, $\frac{144}{169}$.

Find the square root of:

3. $\frac{961}{1089}$. **4.** $\frac{1369}{1681}$. **5.** $\frac{841}{1849}$. **6.** $\frac{2209}{2601}$. **7.** $\frac{2809}{3025}$.

Find the square root, to two decimal places:

8. 2. **9.** 11. **10.** 30. **11.** 125. **12.** $\frac{5}{9}$.

Reduce to fractions with denominators that are perfect squares, and find the square root to two decimal places:

13. $\frac{1}{3}$. **14.** $\frac{2}{5}$. **15.** $\frac{2}{3}$. **16.** $\frac{3}{8}$. **17.** $\frac{7}{8}$.

Find the square root, to two decimal places, of:

18. 0.625. **19.** 0.571. **20.** 0.6. **21.** 0.423. **22.** 0.916.

Find the hypotenuse, given the other sides as follows:

23. 39 ft., 52 ft. **24.** 21 ft., 72 ft. **25.** 51 ft., 68 ft.

Also as follows, carrying the results to three decimals:

26. 82 ft., 35 ft. **27.** 31 ft., 23 ft. **28.** 27 ft., 43 ft.

Given the hypotenuse and one side, find the other side:

29. 10 ft., 6 ft. **30.** 17 in., 15 in. **31.** 25 in., 20 in.

Also as follows, carrying the results to three decimals:

32. 15 ft., 6 ft. **33.** 18 ft., 12 ft. **34.** 23 in., 12 in.

Find the diagonal of the square whose side is:

35. 20″. **36.** 32″. **37.** 45″. **38.** 70″. **39.** 75″.

227. Applications of Square Root. It is proved in geometry that

The corresponding lines of similar figures are to each other as the square roots of the areas of the figures.

That is, if one circle has 16 times the area of another, the radius is 4 times as long as that of the other.

The radius of a circle equals approximately the square root of the quotient of the area of the circle divided by 3.1416.

For example, if the area of a circle is 78.54 sq. in., the number of inches in the radius is $\sqrt{\frac{78.54}{3.1416}} = \sqrt{25} = 5$. Therefore the radius is 5 in.

In Exercise 139 carry the square roots to two decimal places.

Exercise 139. Applications of Square Root

Examples 1 to 3, oral — Examples 4 to 9, written

1. The areas of two squares are 121 sq. in. and 144 sq. in. respectively. What is the ratio of their sides? of their diagonals?

2. The area of one circle is 81 times that of another. What is the ratio of their diameters? of their circumferences?

3. A rectangular lot has a certain frontage and is worth $1000. Find the value of a similar lot which has twice the front and twice the depth.

Find the radius of the circle whose area is:

4. 314.16 sq. in. **5.** 113.0976 sq. in. **6.** 250 sq. in.

7. What must be the diameter of a water main to have the area of a cross section 3 sq. ft.?

8. A horse tethered by a rope can graze over 1570.8 sq. ft. of ground. How long is the rope? If the rope were twice as long, over how much ground could he graze?

9. A tinsmith wishes to make some cylindrical gallon cans. They are to be 10 in. high. What must be the area of the base? What radius must he use to draw it? (1 gal. = 231 cu. in.)

228. The *n*th Power of the *n*th Root. Because $\sqrt{3}$ means one of the two equal factors of 3, $\sqrt{3} \cdot \sqrt{3}$ must equal 3. That is, $(\sqrt{3})^2 = 3$. By the same reasoning

The nth power of the nth root of a number equals the number itself.

That is, $(\sqrt[n]{a})^n = a.$

For a similar reason, *the nth root of the nth power of a number equals the number itself.*

That is, $\sqrt[n]{a^n} = a.$

229. The *n*th root of a Product. Because it has already been proved that

$ab = (\sqrt{a})^2(\sqrt{b})^2,$ § 228

$= (\sqrt{a} \cdot \sqrt{b})^2,$ § 204

therefore $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}.$ Axiom 5

By the same reasoning

The nth root of the product of two or more numbers equals the product of their nth roots.

That is, $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}.$

Reversing this statement, $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$. That is, $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$, $\sqrt[3]{4} \cdot \sqrt[3]{3} = \sqrt[3]{12}$, etc.

230. The *n*th Root of a Power. Since we know that

$\sqrt[3]{a^6} = \sqrt[3]{a^3a^3} = aa = a^2,$ § 229

and that $\sqrt[n]{a^{2n}} = \sqrt[n]{a^na^n} = aa = a^2,$

Therefore, *the exponent of the nth root of any power of a number is found by dividing the exponent of the power by the index of the root.*

That is, $\sqrt[3]{x^{12}} = x^4,\ \sqrt[5]{n^{10}} = n^2.$

Similarly, $\sqrt[n]{a^{mn}} = a^m.$

If the exponent is not exactly divisible by the index of the root, we proceed as follows:

$$\sqrt[5]{n^{11}m^{20}} = \sqrt[5]{n^{10}m^{20}n} = n^2m^4\sqrt[5]{n}.$$

231. Surd. If an indicated root of a positive rational number cannot be obtained exactly, the indicated root is called a *surd*.

For example, $\sqrt{2}$ and $\sqrt[3]{\frac{2}{3}}$ are surds, but any expression in which the radical sign enters is called a *radical*, such as $a + \sqrt{b}$.

232. Surds classified. Surds are classified according to the index of the root, being called *quadratic*, *cubic*, and *biquadratic*, according as this index is 2, 3, or 4.

We might have other names, as *quintic* (fifth root), *sextic* (sixth root), and so on, but it is not necessary to use these terms.

Surds are also classified as *mixed surds*, when there is a rational coefficient; and as *entire surds*, when there is no rational coefficient other than 1.

Thus $2\sqrt{3}$ is a mixed surd, and $\sqrt[3]{7}$ is an entire surd.

233. Reduction of Surds. The changing of the form of a surd without changing its value is called a *reduction* of the surd.

Thus it will be seen that $\sqrt{4a} = 2\sqrt{a}$, and in this case we have reduced $\sqrt{4a}$ to another form.

234. Surds in Simplest Form. When the quantity under the radical sign is integral and as small as possible, a surd is said to be in its *simplest form*.

That is, a number left under the radical sign must be as small as possible, a letter must have as small an exponent as possible, and there must be no fraction left under the radical sign.

For example, $\sqrt{4a}$ is not in its simplest form, for we can extract the square root of 4, nor is $\sqrt[3]{2a^3}$ in its simplest form.

235. Illustrative Problems. Consider the following problems:

1. Simplify $\sqrt[3]{5^3}$, $(\sqrt[3]{5})^3$, and $\sqrt[n]{5^n}$.

$$\sqrt[3]{5^3} = 5,\ (\sqrt[3]{5})^3 = 5, \text{ and } \sqrt[n]{5^n} = 5, \text{ by § 228.}$$

2. Simplify $\sqrt{9a^2b^8}$, $\sqrt[3]{27a^6b^{12}}$, and $\sqrt[5]{32x^{10}}$.

$$\sqrt{9a^2b^8} = 3ab^4,\ \sqrt[3]{27a^6b^{12}} = 3a^2b^4, \text{ and } \sqrt[5]{32x^{10}} = 2x^2, \text{ by §§ 229 and 230}$$

3. Simplify $\sqrt[5]{32a^6x^{10}}$.

$$\sqrt[5]{32a^6x^{10}} = \sqrt[5]{2^5a^5x^{10}a} = 2ax^2\sqrt[5]{a}, \text{ by §§ 229 and 230.}$$

Exercise 140. Radicals

Examples 1 to 4, oral — Examples 5 to 36, written

1. Simplify $\sqrt[3]{a^3}$; $\sqrt[4]{x^4}$; $(\sqrt[5]{n})^5$; $\sqrt[5]{n^5}$; $\sqrt[7]{a^7b^7c^7}$.

2. Simplify $\sqrt{16\,a^2b^{10}}$; $\sqrt[3]{8\,q^6b^{15}}$; $\sqrt[4]{16\,a^{12}b^{20}}$; $\sqrt[n]{a^{2n}b^{mn}}$.

3. Simplify $\sqrt{4\,a^2b}$; $\sqrt[3]{27\,a^6b^2}$; $\sqrt[4]{81x^{12}y^5}$; $\sqrt[5]{32\,x^6y^6}$.

4. Simplify $\sqrt{(a+b)^2(a-b)}$; $\sqrt[4]{(a-b)^2}$; $\sqrt[10]{(x+y)^5}$.

Simplify the following;

5. $\sqrt{196\,a^{12}b^{12}c^{12}}$.

6. $\sqrt{256\,a^{10}b^{16}c^{18}}$.

7. $\sqrt{289\,a^4b^5c^7}$.

8. $\sqrt{361\,x^5y^6z^9}$.

9. $\sqrt{529\,m^2n^8p^9q^{11}}$.

10. $\sqrt[3]{216\,x^3y^3(x+y)^3}$.

11. $\sqrt[3]{27(a+b)^3(c+d)^3}$.

12. $\sqrt[3]{125\,(a+b+c)^4}$.

13. $\sqrt{1849\,(a+b)^2(a-b)^2}$.

14. $\sqrt[3]{1331\,(a-b)^4(c-d)^3}$.

15. $\sqrt[4]{625\,x^5y^5(x-y)^4}$.

16. $\sqrt[4]{(a-b)^4(b-c)^4(c-a)^4}$.

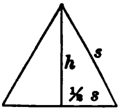
17. $\sqrt[3]{a^4+3\,a^3b+3\,a^2b^2+ab^3}$.

18. $\sqrt[5]{1024\,a^5b^5x^6y^6}$.

19. $\sqrt[5]{ab\,(a+b)^5(x+y)^5}$.

20. $\sqrt{(a^3+2\,a^2b+ab^2)(a^2+ab)}$

21. If the side of an equilateral triangle is 2 in., what is the altitude?



Since h is perpendicular to and bisects the base, therefore $h^2+1^2=2^2$, or $h^2=3$. Therefore $h=\sqrt{3}$.

22. If the side of an equilateral triangle is s, what is the altitude?

As before, h cuts off a right triangle with base $\frac{1}{2}\,s$ and hypotenuse s.

Hence $$h^2+\left(\frac{s}{2}\right)^2=s^2,$$

or $$h^2=s^2-\frac{s^2}{4}=\frac{3\,s^2}{4},$$

and $$h=\sqrt{\frac{3\,s^2}{4}}=\frac{s}{2}\sqrt{3}.$$

23. If the side of an equilateral triangle is $2m$, what is the altitude? the area?

24. If the side of an equilateral triangle is s, what is the area?

25. If the altitude of an equilateral triangle is a, what is the side? the area?

26. If the altitude of an equilateral triangle is 3, what is the side? the area?

27. If the area, a, of an equilateral triangle is $\frac{1}{4}s^2\sqrt{3}$, what is the side? the altitude?

28. If the area of an equilateral triangle is $9\sqrt{3}$ sq. in., what is the side? the altitude?

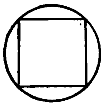
29. If the area of an equilateral triangle is 81 sq. in., find the length of the side to three decimal places.

30. If the area of an equilateral triangle is n sq. in., what is the side? the altitude?

31. From the formula for the area of a circle, $a = \pi r^2$, find the value of r^2. Knowing that $\frac{1}{\pi} = 0.3183$, nearly, find the value of r.

32. If the area of a circle is 50 sq. in., find to three decimal places the value of r.

33. Find the radius of the circle whose area is n sq. in. (Use the value of the reciprocal of π given in Ex. 31.)



34. Find the area of a square inscribed in a circle whose diameter is 10 in.

35. Find to two decimal places the area of a square inscribed in a circle whose circumference is $6\frac{2}{7}$ in.; in a circle whose circumference is 21 in.

36. How many square inches are there in the surface of the largest bar, having a square cross section, that can be cut from a cylinder 6 ft. long and 3 in. in diameter? Find the result to three decimal places.

236. Reduction of a Mixed Surd to an Entire Surd. Since we have found that
$$a^2 = \sqrt[3]{(a^2)^3} = \sqrt[3]{a^6},$$
we see that
$$a^2\sqrt[3]{b} = \sqrt[3]{a^6} \cdot \sqrt[3]{b} = \sqrt[3]{a^6b}.$$

Therefore, *to reduce a mixed surd to an entire surd, raise the coefficient of the surd to the power indicated by the index of the radical, and introduce it as a factor under the radical sign.*

That is,
$$a\sqrt[n]{b} = \sqrt[n]{a^nb}.$$

Reduce $3x\sqrt[3]{xy}$ and $-2a\sqrt{ab}$ to entire surds.
$$3x\sqrt[3]{xy} = \sqrt[3]{(3x)^3xy} = \sqrt[3]{27x^4y}.$$
$$-2a\sqrt{ab} = -\sqrt{(2a)^2ab} = -\sqrt{4a^3b}.$$

Exercise 141. Reduction of a Mixed Surd

Examples 1 to 3, oral — Examples 4 to 33, written

1. Reduce to entire surds: $2\sqrt{a}$; $a\sqrt{a}$; $ab\sqrt[3]{a^2b^2}$.
2. Reduce to entire surds: $x\sqrt[3]{y}$; $x\sqrt[3]{x}$; $(x+y)\sqrt[3]{(x+y)^2}$.
3. Reduce to entire surds: $3a\sqrt{2}$; $3x\sqrt{2xy}$; $5x^2\sqrt{x}$.

Reduce the following mixed surds to entire surds:

4. $2a^2\sqrt{7abc}$.
5. $13p^3\sqrt{2pqr}$.
6. $17mn\sqrt{9mn}$.
7. $7a^2b^2\sqrt[3]{3a^2b^2}$.
8. $5x^2y^3\sqrt[3]{5xy^2}$.
9. $7m^3n\sqrt[4]{m^3n}$.
10. $-11a\sqrt[3]{21a^2}$.
11. $-2x\sqrt[4]{x^2y^3}$.
12. $11pq\sqrt[4]{2p^2q^3}$.
13. $3abc\sqrt[5]{ab^2c^3}$.
14. $-\frac{2}{3}a\sqrt[3]{27a^2b^2}$.
15. $10ab\sqrt[5]{2a^4b^4}$.
16. $2ab\sqrt[7]{ab}$.
17. $3xy\sqrt[4]{x^3y^3}$.
18. $8m\sqrt[3]{4mn^2}$.
19. $2pq\sqrt[3]{p^2q^3}$.
20. $m^2n^2\sqrt[9]{mn}$.
21. $x^2y^3\sqrt[10]{x^3y^2}$.
22. $12ax\sqrt[4]{3a^3x}$.
23. $11mn\sqrt[4]{m^3n^3}$.
24. $a^2b^2c^2\sqrt[5]{a^2b^2c^2}$.
25. $-b^4c^4\sqrt{a^3b^3}$.
26. $-p^2q\sqrt[3]{p^2q}$.
27. $-2axy\sqrt[5]{2axy}$.
28. $3pqr\sqrt[7]{pqr}$.
29. $(a+b)\sqrt{a+b}$.
30. $(a-b)\sqrt{a-b}$.
31. $(x^2+y^2)\sqrt{x}$.
32. $x\sqrt{x^2+y^2}$.
33. $(x+y)\sqrt{x-y}$.

237. The nth Root of a Quotient. If we square $\frac{\sqrt{2}}{\sqrt{3}}$ and $\sqrt{\frac{2}{3}}$ we obtain the same result, thus:

$$\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2 = \frac{\sqrt{2}\cdot\sqrt{2}}{\sqrt{3}\cdot\sqrt{3}} = \frac{2}{3}, \quad \text{and} \quad \left(\sqrt{\frac{2}{3}}\right)^2 = \frac{2}{3}. \qquad \S\ 228$$

We therefore see that $\sqrt{\frac{2}{3}}$ equals $\frac{\sqrt{2}}{\sqrt{3}}$. In like manner,

The nth root of the quotient of two numbers equals the quotient of their nth roots.

Stated in symbols, $$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

For example, $$\sqrt[3]{\frac{27\,a^3b^6c}{125\,x^6y^9}} = \frac{\sqrt[3]{27\,a^3b^6c}}{\sqrt[3]{125\,x^6y^9}} = \frac{3\,ab^2\sqrt[3]{c}}{5\,x^2y^3}.$$

Exercise 142. Roots of Quotients

Examples 1 and 2, oral — Examples 3 to 14, written

1. Simplify $\sqrt{\frac{4}{9}}$; $\sqrt{\frac{25}{36}}$; $\sqrt{\frac{25\,a^4}{49\,x^8y^{10}}}$; $\sqrt{\frac{36\,a^6b^8}{121\,x^4y^{10}}}$; $\sqrt[6]{\frac{a^{12}b}{x^{24}y^6}}$.

2. Simplify $\sqrt[3]{\frac{8}{27}}$; $\sqrt[3]{\frac{27\,a^3}{125\,b^3}}$; $\sqrt[4]{\frac{1}{16}}$; $\sqrt[5]{\frac{a^{10}b^{15}}{x^{15}y^{20}}}$; $\sqrt[3]{-\frac{125\,a^6b^9}{x^9y^{12}z^{15}}}$.

Simplify the following surds:

3. $5\sqrt{\frac{9\,a^2b}{16\,x^2y^4}}$.

4. $7\sqrt[3]{\frac{8\,a^6b}{27\,x^6y^{12}}}$.

5. $8\,ab\sqrt[3]{-\frac{a^3b^5}{x^6y^6}}$.

6. $4\,p^2\sqrt[3]{-\frac{p^9q}{x^{12}y^{15}}}$.

7. $5\,a^2\sqrt[3]{\frac{a^3b^3}{c^3d^3}}$.

8. $2\,xy\sqrt[3]{\frac{1}{8\,x^3y^3}}$.

9. $3\,ab\sqrt[3]{\frac{x^2y^2}{27\,a^6b^6}}$.

10. $2\,m\sqrt[4]{\frac{4\,a^2b^2}{16\,m^4}}$.

11. $\sqrt{\frac{(a+b)^2}{(a-b)^2}}$.

12. $\sqrt{\frac{(a-b)^4}{(a+b)^6}}$.

13. $\sqrt[6]{\frac{(x+2\,y)^2}{(y+2\,x)^6}}$.

14. $\sqrt[5]{\frac{(a+b)^6}{243\,(a-b)^5}}$.

238. The *m*th Root of the *n*th Root. If we separate a number into two equal factors, and each of these factors into three equal factors, we thereby separate the number into six equal factors.

For example, $64 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$. If we separate 64 into two equal factors, each factor equals $2 \cdot 2 \cdot 2$, or 8; and if we separate each of these into three equal factors we have 2 for each factor, and this is one of the six equal factors of 64.

That is, $\sqrt[3]{\sqrt{64}} = \sqrt[3]{8} = 2$, and hence $\sqrt[3]{\sqrt{64}} = \sqrt[6]{64}$.

Therefore the cube root of the square root is the sixth root.

In the same way, *the mth root of the nth root of a number equals the mnth root of the number.*

That is, $$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}.$$

Thus $\sqrt{\sqrt[3]{729}} = \sqrt[6]{729}$, and $\sqrt[3]{\sqrt{729}} = \sqrt[6]{729}$.

Similarly, $\sqrt[6]{49} = \sqrt[3]{\sqrt{49}} = \sqrt[3]{7}$; $\sqrt[10]{32} = \sqrt{\sqrt[5]{32}} = \sqrt{2}$.

Similarly, $$\sqrt[mn]{a^{nx}} = \sqrt[m]{\sqrt[n]{a^{nx}}} = \sqrt[m]{a^x}.$$

Exercise 143. Radicals

Examples 1 to 3, oral — Examples 4 to 15, written

1. Express as a square root: $\sqrt[4]{4}$; $\sqrt[6]{125}$; $\sqrt[6]{8}$; $\sqrt[8]{81}$; $\sqrt[10]{243}$.
2. Express as a cube root: $\sqrt[6]{9}$; $\sqrt[6]{25}$; $\sqrt[9]{8}$; $\sqrt[9]{27}$; $\sqrt[15]{32}$.
3. Express as a single root: $\sqrt{\sqrt{2}}$; $\sqrt{\sqrt[3]{7}}$; $\sqrt[3]{\sqrt{7}}$; $\sqrt[5]{\sqrt[6]{3}}$.

Express as a single root:

4. $\sqrt{\sqrt[3]{a^2+b^2}}$.
6. $\sqrt[n]{\sqrt[2m]{x}}$.
8. $\sqrt[2a-7]{\sqrt[3a+2]{x}}$.
5. $\sqrt[3]{\sqrt{x^2-y^2}}$.
7. $\sqrt[n+7]{\sqrt[2n-3]{3x}}$.
9. $\sqrt[2n-1]{\sqrt[3n+4]{a}}$.

Express as a simpler root:

10. $\sqrt[6]{121}$.
12. $\sqrt[8]{a^4b^6c^8}$.
14. $\sqrt[9]{a^3b^6c^9}$.
11. $\sqrt[6]{1331}$.
13. $\sqrt[10]{32a^{10}b^{15}c^{20}d^{25}}$.
15. $\sqrt[8]{(a+b)^2(a-b)^2}$.

239. Fractional Exponent. The positive integral exponent shows how many times the number which it affects is taken as a factor.

Thus $x^2 = xx$, the exponent showing that x is taken twice as a factor.

There are also fractional exponents, but these must have a different meaning from integral exponents, for we cannot take x half of a time as a factor any more than we can pick up a book half of a time.

Since we have certain laws of positive integral exponents (§ 204), it would be illogical and inconvenient if we should give to the fractional exponent a meaning such that these laws should not also apply to them. Therefore

Such a meaning must be given to fractional exponents as will make the laws of exponents valid for them as well as for positive integral exponents.

Since $a^m \cdot a^m = a^{m+m}$,

we must have $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a.$

But $\sqrt{a} \cdot \sqrt{a} = a.$

We therefore define $a^{\frac{1}{2}}$ to mean $\sqrt{a}$.

And since $(a^m)^n = a^{mn}$,

we must have $(a^{\frac{1}{2}})^3 = a^{\frac{3}{2}}.$

That is, we define $a^{\frac{3}{2}}$ to mean $(\sqrt{a})^3$.

Therefore we see that the following meaning must be given to a fractional exponent if the laws of exponents are to remain valid.

In a fractional exponent the numerator indicates the power of the number affected, and the denominator indicates the root.

Thus $4^{\frac{1}{2}}$ means the same as $\sqrt{4}$ and equals 2;

$8^{\frac{2}{3}}$ means the same as $(\sqrt[3]{8})^2$, or as $\sqrt[3]{8^2}$ and equals 4,

and $a^{\frac{m}{n}}$ means the same as $(\sqrt[n]{a})^m$, or as $\sqrt[n]{a^m}$.

Sometimes it is much easier to use fractional exponents than radicals. In simple cases, like $\sqrt{2}$ or $\sqrt[3]{8}$, it is easier to write the radical sign.

240. Comparison of Fractional Exponents and Radicals. The following comparison of fractional exponents with the ordinary radical sign should be studied with care:

Ordinary Radical Sign		*Fractional Exponent*
$(\sqrt[n]{a})^n = a.$	§ 228	$(a^{\frac{1}{n}})^n = a.$
$\sqrt[n]{a^n} = a.$	§ 230	$(a^n)^{\frac{1}{n}} = a.$
$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}.$	§ 229	$(ab)^{\frac{1}{n}} = a^{\frac{1}{n}}b^{\frac{1}{n}}.$
$\sqrt[n]{a^{mn}} = a^m.$	§ 230	$(a^{mn})^{\frac{1}{n}} = a^m.$
$a\sqrt[n]{b} = \sqrt[n]{a^n b}.$	§ 236	$ab^{\frac{1}{n}} = (a^n b)^{\frac{1}{n}}.$
$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$	§ 237	$\left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}.$
$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}.$	§ 238	$(a^{\frac{1}{n}})^{\frac{1}{m}} = a^{\frac{1}{mn}}.$

241. Reduction of Fractional Exponents. Since the nth root of any power of a number is found by dividing the exponent of the power by the index of the root (§ 230), in the expression $a^{\frac{m}{n}}$ we may divide m by n just as if the exponent $\frac{m}{n}$ were an ordinary fraction. That is,

A fractional exponent may be reduced in the same way as an ordinary fraction.

For example, $a^{\frac{2}{4}} = a^{\frac{1}{2}}$, $x^{\frac{2}{3}} = x^{\frac{4}{6}}$, $m^{\frac{4}{4}} = m$, $(a + b)^{\frac{3}{9}} = (a + b)^{\frac{1}{3}}$, and $(a - b)^{\frac{6}{2}} = (a - b)^3$.

242. Advantages of Fractional Exponents. If we have a complicated expression involving radicals it is usually easier to employ fractional exponents.

For example, $(\sqrt[4]{\sqrt[3]{a^2}})^3 = (a^{\frac{2}{3}})^{\frac{3}{4}} = a^{\frac{2}{3} \cdot \frac{3}{4}} = a^{\frac{1}{2}}.$

Similarly, $(\sqrt[m]{\sqrt[n]{a^p}})^q = \left(a^{\frac{p}{n}}\right)^{\frac{q}{m}} = a^{\frac{pq}{mn}}.$

Exercise 144. Fractional Exponents

Examples 1 to 5, oral — Examples 6 to 56, written

1. What is the meaning of $a^{\frac{1}{2}}$? $a^{\frac{1}{3}}$? $a^{\frac{1}{5}}$? $a^{\frac{1}{6}}$?
2. What is the meaning of $r^{\frac{3}{8}}$? $(abc)^{\frac{2}{3}}$? $a^{\frac{1}{2}}b^{\frac{2}{3}}$? $a^{\frac{3}{4}}b^{\frac{4}{5}}c^{\frac{1}{2}}$?
3. What is the value of $4^{\frac{1}{2}}$? $36^{\frac{1}{2}}$? $81^{\frac{1}{2}}$? $144^{\frac{1}{2}}$? $400^{\frac{1}{2}}$?
4. What is the value of $16^{\frac{3}{4}}$? $32^{\frac{1}{5}}$? $32^{\frac{2}{5}}$? $32^{\frac{4}{5}}$? $125^{\frac{2}{3}}$?
5. What is the value of $8^{\frac{1}{3}} \cdot 32^{\frac{1}{5}}$? $8^{\frac{2}{3}} \cdot 27^{\frac{1}{3}}$? $8^{\frac{2}{3}} \cdot 27^{\frac{2}{3}}$?

Express with radical signs:

6. $a^{\frac{5}{6}}$.
7. $x^{\frac{7}{2}}$.
8. $m^{\frac{3}{8}}$.
9. $2a^{\frac{1}{2}}$.
10. $(2a)^{\frac{1}{2}}$.
11. $2^{\frac{1}{2}}a^{\frac{1}{2}}$.
12. $7x^{\frac{2}{3}}$.
13. $(7x)^{\frac{2}{3}}$.
14. $7^{\frac{2}{3}}x^{\frac{2}{3}}$.
15. $a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}$.
16. $p^{\frac{1}{2}}q^{\frac{2}{3}}r^{\frac{3}{4}}$.
17. $x^{\frac{1}{a}}y^{\frac{1}{b}}z^{\frac{1}{c}}$.

Simplify the following:

18. $81^{\frac{1}{2}}$.
19. $81^{\frac{1}{4}}$.
20. $81^{\frac{3}{4}}$.
21. $125^{\frac{1}{3}}$.
22. $125^{\frac{2}{3}}$.
23. $125^{\frac{4}{3}}$.
24. $-343^{\frac{1}{3}}$.
25. $-343^{\frac{2}{3}}$.
26. $(-32a)^{\frac{1}{5}}$.
27. $(-32a)^{\frac{2}{5}}$.
28. $(-32a)^{\frac{3}{5}}$.
29. $(-32a)^{\frac{4}{5}}$.
30. $32^{\frac{1}{5}} \cdot 27^{\frac{2}{3}}$.
31. $(\frac{1}{25})^{\frac{1}{2}} \cdot (\frac{1}{49})^{\frac{1}{2}}$.
32. $4^{\frac{1}{2}} \cdot (\frac{1}{4})^{\frac{1}{2}} \cdot 9^{\frac{1}{2}}$
33. $125^{\frac{2}{3}} \cdot 625^{\frac{1}{2}}$.
34. $121^{\frac{1}{2}} \cdot 144^{\frac{1}{2}}$.
35. $(\frac{1}{64})^{\frac{1}{2}} \cdot (\frac{1}{1331})^{\frac{2}{3}}$.

Express with fractional exponents:

36. $\sqrt{a}$.
37. $\sqrt[3]{a^2}$.
38. $\sqrt[3]{a^2b}$.
39. $\sqrt[3]{2ab^2}$.
40. $\sqrt[4]{a^3}$.
41. $\sqrt[4]{a^3b}$.
42. $\sqrt[4]{3a^3b^5}$.
43. $\sqrt[5]{a^2b^2}$.
44. $\sqrt[5]{a^2b^3c^4}$.
45. $\sqrt[5]{x^4yz^6}$.
46. $2\sqrt{a}\sqrt[3]{b}$.
47. $3\sqrt[3]{m}\sqrt[4]{n^3}$.
48. $5^2 \cdot \sqrt[3]{m^2}\sqrt[4]{n^5}$.
49. $2a^2\sqrt[4]{p}\sqrt[5]{q^2}$.
50. $\sqrt[6]{a}\sqrt[7]{b}$.
51. $a\sqrt[3]{(mn)^2}$.
52. $\sqrt{(a-b)^3}$.
53. $a\sqrt{b} \cdot c^2\sqrt[3]{d}$.
54. $\sqrt[n]{a} \cdot \sqrt[m]{b}$.
55. $\sqrt[n]{a^m} \cdot \sqrt[p]{b^q}$.
56. $\sqrt[n]{a^{2m}} \cdot \sqrt[p]{b^qc^r}$.

243. Reduction of Surds to a Common Index. From the meaning of the fractional exponent (§ 239) and the fact that such an exponent may be reduced like an ordinary fraction (§ 241), we know that

$$\sqrt[5]{a^4} = a^{\frac{4}{5}} = a^{\frac{12}{15}} = \sqrt[15]{a^{12}},$$

and that

$$\sqrt[3]{a^2} = a^{\frac{2}{3}} = a^{\frac{10}{15}} = \sqrt[15]{a^{10}}.$$

In the same way we may reduce any two surds to surds having the same index.

That is, $$\sqrt[m]{a^k} = a^{\frac{k}{m}} = a^{\frac{kn}{mn}} = \sqrt[mn]{a^{kn}}.$$

Reduce $\sqrt[3]{4}$ and $\sqrt[4]{27}$ to surds having a common index.

$$\sqrt[3]{4} = \sqrt[3]{2^2} = 2^{\frac{2}{3}};$$
$$\sqrt[4]{27} = \sqrt[4]{3^3} = 3^{\frac{3}{4}}.$$

Since the L.C.D. of $\frac{2}{3}$ and $\frac{3}{4}$ is 12, we reduce to twelfths.

Then $$\sqrt[3]{4} = 2^{\frac{2}{3}} = 2^{\frac{8}{12}} = \sqrt[12]{2^8};$$
$$\sqrt[4]{27} = 3^{\frac{3}{4}} = 3^{\frac{9}{12}} = \sqrt[12]{3^9}.$$

Exercise 145. Reduction of Surds to a Common Index

Examples 1 to 3, oral — Examples 4 to 16, written

1. Express as a twelfth root: $a^{\frac{1}{2}}$; $b^{\frac{1}{3}}$; $c^{\frac{1}{4}}$; $d^{\frac{1}{6}}$; $\sqrt{x}$; $\sqrt[3]{y}$; $\sqrt[4]{z}$.

2. Express as a tenth root: $x^{\frac{1}{2}}$; $y^{\frac{1}{5}}$; $\sqrt{z}$; $\sqrt[5]{w}$; $\sqrt[5]{m^2}$; $\sqrt[5]{n^3}$.

3. Express $x^{\frac{1}{3}}$ and $y^{\frac{3}{4}}$ with a common root index.

Reduce to surds having a common root index:

4. $\sqrt{5}, \sqrt[3]{2}$.

5. $\sqrt[3]{2}, \sqrt[4]{3}$.

6. $\sqrt[4]{a^3}, \sqrt[5]{b^3}$.

7. $\sqrt[3]{p^2}, \sqrt[4]{q^3}$.

8. $a^{\frac{1}{2}}, b^{\frac{2}{3}}$.

9. $x^{\frac{1}{3}}, \sqrt{y}$.

10. $2a^{\frac{2}{3}}, \sqrt[4]{a}$.

11. $3x^{\frac{3}{4}}, \sqrt{x}$.

12. $2\sqrt{2}, 3\sqrt[3]{2}, 2\sqrt[4]{3}$.

13. $3\sqrt{2}, 5\sqrt[3]{3}, 3\sqrt[4]{7}$.

14. $2\sqrt{2}, 3\sqrt[4]{3}, 5\sqrt[6]{4}$.

15. $a^{\frac{2}{3}}, b^{\frac{1}{4}}, c^{\frac{5}{6}}, \sqrt{d}$.

16. Reduce to surds having a common index, and then arrange in order of magnitude: $\sqrt{8}$, $\sqrt[3]{10}$, $\sqrt[6]{81}$; $\sqrt{3}$, $\sqrt[3]{5}$, $\sqrt[6]{23}$; $\sqrt[3]{7}$, $\sqrt[4]{15}$, $\sqrt{2}$.

244. Similar Surds. Surds which, when reduced to their simplest forms, have the same surd factor are called ***similar surds.***

For example, $3\sqrt{2}$, $4\sqrt{2}$, and $9\sqrt{2}$ are similar surds.

245. Addition and Subtraction of Surds. The principles of addition and subtraction of similar surds are the same as those of other algebraic expressions.

Thus $$3\sqrt{2}+4\sqrt{2}=7\sqrt{2},$$

and $$a\sqrt[n]{x}+b\sqrt[n]{x}-c\sqrt[n]{x}=(a+b-c)\sqrt[n]{x}.$$

Simplify $\sqrt{243}-\sqrt{75}$.

$$\begin{aligned}\sqrt{243}&=\sqrt{81\cdot 3}=9\sqrt{3};\\ \sqrt{75}&=\sqrt{25\cdot 3}=5\sqrt{3}.\\ \sqrt{243}-\sqrt{75}&=9\sqrt{3}-5\sqrt{3}=4\sqrt{3}.\end{aligned}$$

Exercise 146. Addition and Subtraction of Surds

Examples 1 to 3, oral — Examples 4 to 12, written

1. Simplify $3\sqrt{a}+9\sqrt{a}$; $17\sqrt[3]{x^2}-9\sqrt[3]{x^2}$.
2. Simplify $39\sqrt[3]{4}+17\sqrt[3]{4}$; $41\sqrt[5]{3}-16\sqrt[5]{3}$.
3. Simplify $41\sqrt[5]{-a^2b}+17\sqrt[5]{-a^2b}$; $43\sqrt[7]{x^2y^3}-19\sqrt[7]{x^2y^3}$.

Simplify the following:

4. $2\sqrt{x}+3\sqrt{x}+9\sqrt{x}-4\sqrt{x}+2\sqrt{x}-6\sqrt{x}$.
5. $3\sqrt{a}-7\sqrt{b}+4\sqrt{a}-2\sqrt{b}+5\sqrt{a}-\sqrt{b}$.
6. $3\sqrt{4m}-2\sqrt{9m}+2\sqrt{25m}-4\sqrt{49m}+12\sqrt{81m}$.
7. $5\sqrt{9x}+7\sqrt[3]{2x}-3\sqrt{25x}+4\sqrt[3]{16x}+10\sqrt{100x}$.
8. $3\sqrt{8}+2\sqrt{32}+7\sqrt{50}-6\sqrt{162}+9\sqrt{98}+7\sqrt{242}$.
9. $\sqrt{(a+b)^2c}+\sqrt{(a-b)^2c}+\sqrt{a^2c}-\sqrt{b^2c}+(a+b)c$.
10. $99^{\frac{1}{2}}+44^{\frac{1}{2}}+176^{\frac{1}{2}}+275^{\frac{1}{2}}-396^{\frac{1}{2}}+2\sqrt{11}$.
11. $\sqrt[3]{16}+\sqrt[3]{54}+\sqrt[3]{2662}-\sqrt[3]{128}+\sqrt[3]{-250}$.
12. $\sqrt[4]{32}+2\sqrt[4]{162}+3\sqrt[4]{512}+5\sqrt[4]{1250}-\frac{1}{6}\sqrt[4]{2592}$.

246. Multiplication of Surds. Since we have found (§ 229) that $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$, therefore

To multiply one surd by another, express the surds with a common index. Then find the product of the coefficients for the required coefficient, and the product of the surd factors for the required surd factor.

In all work with radicals the results should be reduced to their simplest forms.

Simplify $4\sqrt{3} \cdot 2\sqrt[3]{9}$.

$$\begin{aligned} 4\sqrt{3} \cdot 2\sqrt[3]{9} &= 4\sqrt[6]{3^3} \cdot 2\sqrt[6]{9^2} = 4\sqrt[6]{3^3} \cdot 2\sqrt[6]{3^4} \\ &= 8\sqrt[6]{3^7} = 8 \cdot 3\sqrt[6]{3} = 24\sqrt[6]{3}. \end{aligned}$$

We may also express the numbers with fractional exponents, thus:

$$\begin{aligned} 4 \cdot 3^{\frac{1}{2}} \cdot 2 \cdot 9^{\frac{1}{3}} &= 4 \cdot 2 \cdot 3^{\frac{1}{2}} \cdot 3^{\frac{2}{3}} = 8 \cdot 3^{\frac{1}{2}+\frac{2}{3}} \\ &= 8 \cdot 3^{\frac{7}{6}} = 8 \cdot 3 \cdot 3^{\frac{1}{6}} = 24 \cdot 3^{\frac{1}{6}}. \end{aligned}$$

Exercise 147. Multiplication of Radicals

Examples 1 to 3, oral — Examples 4 to 20, written

1. Simplify: $a^{\frac{1}{4}}a^{\frac{1}{2}}$; $a^{\frac{1}{2}}a^{\frac{3}{4}}$; $a^{\frac{1}{2}}a^{\frac{1}{3}}$; $\sqrt{x} \cdot \sqrt[3]{x}$.
2. Simplify: $a^{\frac{1}{3}}a^{\frac{2}{3}}$; $\sqrt[3]{x} \cdot \sqrt[3]{x^2}$; $a^{\frac{1}{3}}a^{\frac{1}{6}}$; $\sqrt[3]{a} \cdot \sqrt[6]{a^5}$.
3. Simplify: $\sqrt{3} \cdot \sqrt{12}$; $\sqrt{2} \cdot \sqrt{18}$; $\sqrt{5} \cdot \sqrt{20}$; $\sqrt{3} \cdot \sqrt{27}$.
4. $\sqrt{20} \cdot \sqrt{30}$.
5. $\sqrt[3]{5} \cdot \sqrt[3]{135}$.
6. $\sqrt[3]{a} \cdot \sqrt[5]{a^2}$.
7. $2\sqrt{7} \cdot 7\sqrt{2}$.
8. $\sqrt{18} \cdot \sqrt[3]{6}$.
9. $2\sqrt{3} \cdot 3\sqrt[3]{2}$.
10. $4\sqrt{5} \cdot 2\sqrt[3]{3}$.
11. $\sqrt[3]{9} \cdot 27\sqrt{3}$.
12. $\sqrt[3]{16} \cdot 5\sqrt[3]{4}$.
13. $\sqrt[3]{-16} \cdot \sqrt{18}$.
14. $\sqrt{6} \cdot \sqrt{864}$.
15. $\sqrt[3]{9} \cdot \sqrt{800}$.
16. $(\sqrt{2} + 3\sqrt{18} + 9\sqrt{50} - 7\sqrt{32})\sqrt{2}$.
17. $(\sqrt[3]{24} + 8\sqrt[3]{81} - 2\sqrt[3]{375} - 3\sqrt[3]{192})\sqrt[3]{3}$.
18. $(\sqrt{12} + 2\sqrt{27} + 8\sqrt{75} - 5\sqrt{108} + 4\sqrt{147} + \sqrt{192})\sqrt{3}$.
19. $(\sqrt{5} + \sqrt{20} + 9\sqrt{45} - 2\sqrt{80} - 3\sqrt{125} + 5\sqrt{180})\sqrt{5}$.
20. $(\sqrt[3]{2} + 5\sqrt[3]{54} - 6\sqrt[3]{128} + 5\sqrt[3]{16} - 2\sqrt[3]{250} + 7\sqrt[3]{432})\sqrt[3]{2}$.

247. Multiplication of Polynomial Radicals. Polynomial radicals are multiplied like ordinary polynomials. For example,

$$\begin{array}{rrrl}
3\sqrt{5} & +\ 2\sqrt{3} & & \\
7\sqrt{5} & -\ 4\sqrt{3} & & \\ \hline
105 & +14\sqrt{15} & & \\
 & -12\sqrt{15} & -24 & \\ \hline
105 & +\ 2\sqrt{15} & -24 & = 81+2\sqrt{15}.
\end{array}$$

Exercise 148. Multiplication of Polynomial Radicals

Examples 1 to 4, oral — Examples 5 to 24, written

1. Multiply $2\sqrt{3}+3\sqrt{2}$ by 2; by 3; by 5; by -4.
2. Multiply $\sqrt{2}+1$ by $\sqrt{2}-1$; by $\sqrt{2}+1$.
3. Multiply $\sqrt{2}+\sqrt{3}$ by $\sqrt{2}-\sqrt{3}$; by $\sqrt{2}+\sqrt{3}$.
4. Square $\sqrt{5}+\sqrt{2}$; $\sqrt{5}-\sqrt{2}$; $\sqrt{2}+\sqrt{5}$; $\sqrt{2}-\sqrt{5}$.

5. $(\sqrt{8}+\sqrt{5})(\sqrt{8}-\sqrt{5})$.
6. $(\sqrt{8}+\sqrt{7})(\sqrt{8}+2\sqrt{7})$.
7. $(\sqrt{10}-\sqrt{3})(\sqrt{10}+4\sqrt{3})$.
8. $(7-3\sqrt{6})(9+7\sqrt{6})$.
9. $(a+2\sqrt{b})(3a-4\sqrt{b})$.
10. $(x-3\sqrt{y})(2x+7\sqrt{y})$.
11. $\sqrt{7+\sqrt{18}}\cdot\sqrt{7-\sqrt{18}}$.
12. $\sqrt{7+2\sqrt{6}}\cdot\sqrt{7-2\sqrt{6}}$.
13. $\sqrt{9+\sqrt{17}}\cdot\sqrt{9-\sqrt{17}}$.
14. $(3+\sqrt[3]{2})(7-2\sqrt[3]{2})$.
15. $(4+\sqrt[4]{3})(5-2\sqrt[4]{3})$.
16. $(\sqrt[5]{2}+\sqrt[5]{3})(\sqrt[5]{2}-\sqrt[5]{3})$.
17. $(12\sqrt{5}-11\sqrt{3})(13\sqrt{5}+19\sqrt{3})$.
18. $(5+9\sqrt{3}+6\sqrt{5})(8-7\sqrt{3}-9\sqrt{5})$.
19. $(2+7\sqrt{6}-12\sqrt{5})(7-3\sqrt{6}+8\sqrt{5})$.
20. $(2\sqrt{7}-8\sqrt{28}+5\sqrt{63})(2\sqrt{7}-8\sqrt{28}-5\sqrt{63})$.
21. $(\sqrt[3]{9}-7\sqrt[3]{72}+6\sqrt[3]{1125})(3\sqrt[3]{2}+8\sqrt[3]{3}-4\sqrt[3]{9})$.
22. $(\sqrt{a}+\sqrt{b})(a^2+ab+b^2)(\sqrt{a}-\sqrt{b})$.
23. $(\sqrt{x}-\sqrt{y})(x+y)(\sqrt{x}+\sqrt{y})(x^2+y^2)$.
24. $(a-\sqrt{ab}+b)(a+\sqrt{ab}+b)(a^2-ab+b^2)$.

248. Division of Radicals. If, in the solution of a problem, we find $\frac{3}{\sqrt{5}}$ as the result, we have an awkward numerical expression. If we extract the square root of 5 to any number of decimal places, we have a long divisor. But if we multiply both terms by $\sqrt{5}$, we have $\frac{3\sqrt{5}}{5}$, and the division is much simpler.

Comparing the two operations we have:

$$(1)\quad \frac{3}{\sqrt{5}} = \frac{3}{2.236} = 1.3416, \text{ nearly.}$$

$$(2)\quad \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5} = \frac{6.708}{5} = 1.3416, \text{ nearly.}$$

The work in (1) will be found more difficult than that in (2).

Therefore, *if the denominator of a fraction contains a surd, multiply both terms of the fraction by a number that will make the denominator rational.*

This is called *rationalizing the denominator*, the expression meaning that the denominator is to be made rational without changing the value of the fraction.

1. Divide 3 by $\sqrt{2}$.

$\frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$. The result should be left in this form unless the extraction of the root is required.

2. Divide $\sqrt{2}$ by $\sqrt{3}$.

$$\frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{2}\,.\,\sqrt{3}}{3} = \frac{1}{3}\sqrt{6}.$$

3. Simplify $\sqrt[3]{\frac{3}{4}}$.

$$\sqrt[3]{\tfrac{3}{4}} = \sqrt[3]{\tfrac{6}{8}} = \tfrac{1}{2}\sqrt[3]{6}.$$

4. Divide $7\sqrt[3]{50}$ by $\sqrt{2}$.

$$\frac{7\sqrt[3]{50}}{\sqrt{2}} = \frac{7\sqrt{2}\sqrt[3]{50}}{2} = \frac{7\sqrt[6]{8\cdot 2500}}{2} = \frac{7}{2}\sqrt[6]{20{,}000}.$$

Exercise 149. Division of Radicals

Examples 1 to 4, oral — Examples 5 to 28, written

1. Simplify $\frac{1}{\sqrt{2}}$; $\frac{2}{\sqrt{2}}$; $\frac{1}{\sqrt[3]{2^2}}$; $\frac{1}{\sqrt[3]{3^2}}$; $\frac{2}{\sqrt{5}}$; $\frac{3}{\sqrt{7}}$.

2. Simplify $\sqrt{\frac{2}{3}}$; $\sqrt{\frac{3}{5}}$; $\sqrt[3]{\frac{3}{4}}$; $\sqrt[3]{\frac{5}{9}}$; $\sqrt[3]{\frac{3}{5^2}}$; $\sqrt[3]{\frac{7}{25}}$.

3. Simplify $\sqrt[4]{\frac{2}{3^3}}$; $\sqrt[4]{\frac{5}{27}}$; $\sqrt[5]{\frac{3}{2^4}}$; $\sqrt[5]{\frac{5}{16}}$; $\sqrt[6]{\frac{2}{3^5}}$.

4. Simplify $\sqrt{a^3} \div \sqrt{a}$; $\sqrt{a^5} \div \sqrt{a^3}$; $\sqrt{ab} \div \sqrt{a}$.

Divide:

5. $\sqrt[3]{63}$ by $\sqrt[3]{3^2}$.

6. $\sqrt[3]{3^2}$ by $\sqrt[3]{2^2}$.

7. $75^{\frac{1}{5}}$ by $5^{\frac{1}{5}}$.

8. $25^{\frac{3}{5}}$ by $5^{\frac{3}{5}}$.

9. $\sqrt{\frac{2}{3}}$ by $\sqrt{\frac{3}{5}}$.

10. $\sqrt[3]{1\frac{1}{2}}$ by $\sqrt[3]{\frac{3}{16}}$.

11. 7 by $\sqrt{21}$.

12. 9 by $\sqrt{21}$.

13. $\sqrt{a}$ by $\sqrt[3]{a}$.

14. $\sqrt[3]{a^2}$ by $\sqrt{a}$.

15. ab^4 by $\sqrt{ab}$.

16. xy by $\sqrt[3]{x^2y^2}$.

17. From $a\sqrt{x} - b\sqrt{x} + c\sqrt{x}$ remove the factor $\sqrt{x}$. What is the quotient of $a\sqrt{x} - b\sqrt{x} + c\sqrt{x}$ divided by $\sqrt{x}$?

Divide:

18. $8\sqrt{50} - 7\sqrt{32} + 2\sqrt{18} - 6\sqrt{98}$ by $2\sqrt{2}$.

19. $9\sqrt[3]{24} + 6\sqrt[3]{81} - 3\sqrt[3]{192} + 12\sqrt[3]{375}$ by $3\sqrt[3]{3}$.

20. $56\sqrt{30} - 84\sqrt{10} + 100\sqrt{15}$ by $4\sqrt{35}$.

21. $10\sqrt{14} - 5\sqrt{63} + 4\sqrt{28} + 6\sqrt{7}$ by $2\sqrt{7}$.

22. $6\sqrt{33} + 5\sqrt{22} - 7\sqrt{66} + 8\sqrt{44}$ by $\sqrt{11}$.

23. $a\sqrt[3]{a^2b^5} - b\sqrt[3]{a^5b^2} + c\sqrt[3]{a^5b^5} - d\sqrt[3]{a^3b^2}$ by $\sqrt[3]{a^2b^2}$.

24. $a^{\frac{1}{2}}b^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{2}} - a^{\frac{2}{3}}b^{\frac{1}{2}} + a^{\frac{1}{2}}b^{\frac{1}{3}}$ by $a^{\frac{1}{3}}b^{\frac{1}{3}}$.

25. $4\sqrt{15} + 16\sqrt{35} - 10\sqrt{30} + 12\sqrt{50}$ by $2\sqrt{5}$.

26. $9\sqrt[3]{35} + 15\sqrt[3]{45} + 30\sqrt[3]{55} - 21\sqrt[3]{60}$ by $3\sqrt[3]{5}$.

27. $abc\sqrt[3]{abc} + a^2bc\sqrt[3]{a^2bc} - ab^2c\sqrt[3]{ab^2c} + \sqrt[3]{abc^2}$ by $\sqrt[3]{abc}$.

28. $(a+b)\sqrt{a+b} - (a-b)\sqrt{a^2-b^2} + ab\sqrt{a^2b+ab^2}$ by $\sqrt{a+b}$.

249. Division by a Binomial Surd. If the divisor or the denominator of a fraction is a binomial involving only quadratic surds, it may be rationalized by the process shown in the following example:

Simplify $\dfrac{2+3\sqrt{2}}{3-5\sqrt{2}}$.

Multiply both terms of the fraction by $3+5\sqrt{2}$, because the denominator will then become the difference of two squares. Then

$$\frac{2+3\sqrt{2}}{3-5\sqrt{2}}=\frac{(3+5\sqrt{2})(2+3\sqrt{2})}{(3+5\sqrt{2})(3-5\sqrt{2})}=\frac{36+19\sqrt{2}}{9-50}=-\frac{1}{41}(36+19\sqrt{2}).$$

The binomial $3+5\sqrt{2}$ is called the *conjugate* of $3-5\sqrt{2}$. In such cases we multiply both terms of the fraction by the conjugate of the denominator, thus rationalizing the denominator.

Exercise 150. Division by a Binomial Surd

Examples 1 to 3, oral — Examples 4 to 36, written

1. What is the conjugate of $3-\sqrt{7}$? of $4+\sqrt{5}$?

2. By what should $\sqrt{7}+\sqrt{5}$ be multiplied that the product may be rational? What is the product?

3. Multiply $12+\sqrt{10}$ by $12-\sqrt{10}$; $\sqrt{51}+\sqrt{11}$ by $\sqrt{51}-\sqrt{11}$; $9\sqrt{2}-\sqrt{10}$ by $9\sqrt{2}+\sqrt{10}$.

Divide:

4. $\sqrt{3}$ by $\sqrt{5}-\sqrt{2}$.
5. 5 by $\sqrt{3}-\sqrt{2}$.
6. $\sqrt{2}$ by $\sqrt{5}-\sqrt{2}$.
7. $3+\sqrt{2}$ by $3-\sqrt{2}$.
8. $5+\sqrt{7}$ by $\sqrt{7}-5$.
9. $\sqrt{3}+\sqrt{2}$ by $\sqrt{3}-\sqrt{2}$.
10. $8+7\sqrt{5}$ by $2-3\sqrt{5}$.
11. $3+\sqrt{6}$ by $\sqrt{2}+\sqrt{3}$.
12. $5\sqrt{3}-3\sqrt{5}$ by $\sqrt{5}-\sqrt{3}$.
13. $7\sqrt{5}+5\sqrt{7}$ by $\sqrt{5}+\sqrt{7}$.
14. $2\sqrt{3}+\sqrt{6}$ by $\sqrt{3}+\sqrt{6}$.
15. $a+b\sqrt{x}$ by $a-b\sqrt{x}$.

16. Simplify $\dfrac{3+\sqrt{5}}{3-\sqrt{5}}$ and find the approximate value to two decimal places.

Rationalize the denominators:

17. $\dfrac{2+\sqrt{3}}{2-\sqrt{3}}$.

18. $\dfrac{2}{7-3\sqrt{5}}$.

19. $\dfrac{4-2\sqrt{2}}{4+2\sqrt{2}}$.

20. $\dfrac{5-3\sqrt{3}}{5+3\sqrt{3}}$.

21. $\dfrac{2+\sqrt{2}}{3-\sqrt{2}}$.

22. $\dfrac{4+\sqrt{7}}{5-\sqrt{7}}$.

23. $\dfrac{5-\sqrt{2}}{5+\sqrt{3}}$.

24. $\dfrac{2-\sqrt{5}}{3+\sqrt{2}}$.

25. $\dfrac{a+\sqrt{b}}{a-\sqrt{b}}$.

26. $\dfrac{a-\sqrt{b}}{a+\sqrt{b}}$.

27. $\dfrac{a+\sqrt{b}}{c-\sqrt{d}}$.

28. $\dfrac{x+y\sqrt{y}}{x-y\sqrt{y}}$.

29. Simplify $\dfrac{3+\sqrt{7}}{3-\sqrt{7}}$, and find the approximate value to three decimal places.

30. Simplify $\dfrac{5+2\sqrt{3}}{5-2\sqrt{3}}$, and find the approximate value to three decimal places.

31. Arrange in order of magnitude, beginning with the largest: $\dfrac{3+\sqrt{7}}{3-\sqrt{7}}$, $\dfrac{3+\sqrt{5}}{3-\sqrt{5}}$, and $\dfrac{4+\sqrt{6}}{4-\sqrt{6}}$.

32. In the fraction $\dfrac{2+\sqrt{3}+\sqrt{5}}{2+\sqrt{3}-\sqrt{5}}$, multiply both terms by $2+\sqrt{3}+\sqrt{5}$. Then rationalize the denominator of the resulting fraction.

33. Rationalize the denominator of the fraction $\dfrac{3+\sqrt{2}+\sqrt{3}}{3-\sqrt{2}+\sqrt{3}}$ by a method similar to that of Ex. 32.

34. Rationalize the denominator of the fraction $\dfrac{5+\sqrt{2}-\sqrt{3}}{5+\sqrt{2}+\sqrt{3}}$ by a method similar to that of Ex. 32.

35. Rationalize the denominator of the fraction $\dfrac{7+\sqrt{3}-\sqrt{2}}{7+\sqrt{3}+\sqrt{2}}$.

36. Rationalize the denominator and then find, to three decimal places, the approximate value of $\dfrac{3.5\sqrt{2}+\frac{1}{2}\sqrt{6}-1}{3.5\sqrt{2}+\frac{1}{2}\sqrt{6}+1}$.

250. Powers of Radicals. The powers of radicals are conveniently found by the following methods:

1. Find the square of $2\sqrt[3]{5^2}$ and the cube of $5\sqrt{x}$.

$$(2\sqrt[3]{5^2})^2 = (2 \cdot 5^{\frac{2}{3}})^2 = 2^2 \cdot 5^{\frac{4}{3}} = 4 \cdot 5 \cdot 5^{\frac{1}{3}} = 20\sqrt[3]{5}.$$

$$(5\sqrt{x})^3 = (5x^{\frac{1}{2}})^3 = 125x^{\frac{3}{2}} = 125xx^{\frac{1}{2}} = 125x\sqrt{x}.$$

2. Find the square of $2 + \sqrt{3}$.

$$\begin{aligned}(2+\sqrt{3})^2 &= 4 + 4\sqrt{3} + 3\\ &= 7 + 4\sqrt{3}.\end{aligned}$$

Exercise 151. Powers of Radicals

Examples 1 to 4, oral — Examples 5 to 30, written

1. Raise to the second power: $\sqrt{7}$; $\sqrt[4]{3}$; $\sqrt{ab}$; $\sqrt[6]{a}$.
2. Raise to the third power: $\sqrt[3]{x}$; $\sqrt[3]{x^2}$; $x^{\frac{1}{6}}$; $x^{\frac{5}{6}}$.
3. Raise to the fourth power: $\sqrt{a}$; $a^{\frac{1}{4}}$; $a^{\frac{3}{4}}$; $\sqrt[4]{m^3}$.
4. Raise to the fifth power: $\sqrt[5]{2x}$; $x^{\frac{3}{5}}$; $\sqrt[5]{m^3}$; $\sqrt[5]{x^4y^4}$.

Perform the operations indicated:

5. $(\sqrt[3]{a^2})^2$.
6. $(\sqrt[3]{a^2})^4$.
7. $(\sqrt[4]{x^3})^5$.
8. $(\sqrt[4]{x^2y^3})^5$.
9. $(\sqrt[5]{a^2b^3c^4})^6$.
10. $(\sqrt[3]{-a^2b^2})^4$.
11. $(-\sqrt[3]{a^2b^2})^4$.
12. $(-\sqrt[4]{111})^3$.
13. $(\sqrt[6]{x})^3$.
14. $(\sqrt[6]{x})^4$.
15. $(\sqrt[6]{x})^8$.
16. $\sqrt[3]{\sqrt{a}}$.
17. $\sqrt[3]{\sqrt[4]{a^3}}$.
18. $\sqrt[5]{\sqrt[3]{x^5}}$.
19. $\sqrt[4]{\sqrt{x^3}}$.
20. $\sqrt[3]{\sqrt[4]{a^{12}}}$.
21. $[(a+b)^{\frac{1}{2}}]^4$.
22. $[\sqrt{(a+b)^3}]^3$.
23. $(a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}})^{12}$.
24. $(a^{\frac{1}{2}}b^{\frac{2}{3}}c^{\frac{3}{4}})^{12}$.
25. $[(1+x^2)^{\frac{1}{2}}]^6$.
26. $[(\sqrt{a-b})^{\frac{1}{2}}]^4$.
27. $[(\sqrt[3]{a+b})^{\frac{1}{2}}]^6$.
28. $[(\sqrt[5]{x+y})^{\frac{1}{3}}]^{30}$.
29. Multiply $(\sqrt[3]{4})^2$ by $\sqrt[3]{4}$; by $\sqrt[3]{4^2}$; by $(\sqrt[3]{4})^4$; by $\sqrt{4^3}$.
30. Divide $(\sqrt[4]{a^3b^3})^6$ by $\sqrt[4]{a^3b^3}$; by $(\sqrt[4]{a^3b^3})^3$; by $(\sqrt[6]{a^5b^5})^{12}$.

251. Negative Exponents. We have found the meaning of positive integral exponents as follows:

a^3 means aaa, a^n means $aaa \cdots$ (n factors).

We have also found the meaning of fractional exponents as follows:

$a^{\frac{2}{3}}$ means $\sqrt[3]{a^2}$ or $\left(\sqrt[3]{a}\right)^2$, $a^{\frac{m}{n}}$ means $\sqrt[n]{a^m}$ or $\left(\sqrt[n]{a}\right)^m$.

We also have occasion to use negative exponents in algebra, and their meaning will now be considered. As with fractional exponents,

Such a meaning must be given to negative exponents as will make the laws of exponents valid for them as well as for positive integral exponents.

Since $a^m \div a^n = a^{m-n}$, we know that $a^5 \div a^2 = a^3$. We must therefore have $a^2 \div a^5 = a^{2-5} = a^{-3}$. But $a^2 \div a^5 = \frac{1}{a^3}\cdot$ We must therefore have $a^{-3} = \frac{1}{a^3}\cdot$

More generally, we must have $a^m \cdot a^{-n} = a^{m-n}$; but $a^m \cdot \frac{1}{a^n} = a^{m-n}$, and hence a^{-n} must equal $\frac{1}{a^n}\cdot$

Therefore, *a quantity affected by a negative exponent equals the reciprocal of that quantity affected by a numerically equal positive exponent.*

That is, $$a^{-n} = \frac{1}{a^n}.$$

Hence
$$a^{-2} = \frac{1}{a^2}, \qquad a^{-\frac{1}{2}} = \frac{1}{a^{\frac{1}{2}}} = \frac{1}{\sqrt{a}},$$
$$a^{-3} = \frac{1}{a^3}, \qquad a^{-\frac{2}{3}} = \frac{1}{a^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{a^2}}.$$

If we have a complicated expression involving radicals and fractions it is usually easier to employ negative and fractional exponents than to use positive exponents and radical signs.

For example, $\sqrt[5]{\dfrac{\sqrt{a^3}}{\sqrt[3]{a^2}}} = \left(a^{\frac{3}{2}} \cdot a^{-\frac{2}{3}}\right)^{\frac{1}{5}} = \left(a^{\frac{3}{2}-\frac{2}{3}}\right)^{\frac{1}{5}} = \left(a^{\frac{5}{6}}\right)^{\frac{1}{5}} = a^{\frac{1}{6}}$, or $\sqrt[6]{a}$.

252. Zero as an Exponent. It now remains to find the meaning of zero as an exponent. If a^0 is to conform to the same laws as a^m, we must have $a^0 \cdot a^m = a^{0+m} = a^m$. But to have this true, a^0 must equal 1. Therefore

A quantity affected by an exponent zero is equal to 1.

That is, $$x^0 = 1.$$

For example, $2^0 = 1$, $(\frac{1}{2})^0 = 1$, $(-1)^0 = 1$.

Exercise 152. Negative and Zero Exponents

Examples 1 to 22, oral — Examples 23 to 31, written

1. Express a^{-5}, using a positive exponent.

2. What is the value of 2^0? of $(-3)^0$? of $(\frac{1}{2})^0$? of a^0? of 4^0? of 4^{-1}? of $4^{\frac{1}{2}}$? of $4^{\frac{3}{2}}$?

Express with positive exponents:

3. a^{-7}.
4. x^{-9}.
5. $(\frac{1}{2})^{-1}$.
6. $(\frac{3}{4})^{-1}$.
7. $(\frac{1}{2})^{-2}$.
8. $(\frac{1}{3})^{-3}$.
9. $8^{-\frac{1}{3}}$.
10. $8^{-\frac{2}{3}}$.
11. $27^{-\frac{2}{3}}$.
12. $a^{-\frac{2}{3}}$.
13. $(a+b)^{-1}$.
14. $a^{-1}+b^{-1}$.

Express in integral form, using negative and fractional exponents:

15. $\frac{1}{2}$.
16. $\frac{1}{a}$.
17. $\frac{2}{3}$.
18. $\frac{2}{a^2}$.
19. $\frac{1}{25}$.
20. $\frac{a}{\sqrt{b}}$.
21. $\frac{1}{81}$.
22. $\frac{a}{b\sqrt{c}}$.

Copy and express in integral form, using negative and fractional exponents:

23. $\frac{ab}{xy}$.
24. $\frac{ab}{\sqrt{xy}}$.
25. $\frac{pq}{\sqrt[3]{a^2b^2}}$.
26. $\frac{a+b}{a-b}$.
27. $\frac{a+b}{\sqrt{a-b}}$.
28. $\frac{a-b}{\sqrt[3]{(a+b)^2}}$.
29. $\frac{1}{a^2+2ab+b^2}$.
30. $\frac{1}{\sqrt[3]{a^2-2ab+b^2}}$.
31. $\frac{1}{a^2+b^2}(a^2-b^2)$.

253. Operations involving Negative, Zero, and Fractional Exponents. Since we have defined negative, zero, and fractional exponents so that they conform to the ordinary laws of exponents, we operate with them in the same way as with positive integral exponents.

In the following operations the quantities having negative exponents are compared with the fractional forms to show their relation. In solving the problems in Exercise 153, however, use negative exponents whenever they appear, without changing to fractional forms.

1. Multiply $x^{-1}+2$ by $x^{-1}-7$.

$$\begin{aligned}&(x^{-1}+2)(x^{-1}-7)\\&\quad=x^{-2}+2x^{-1}-7x^{-1}-14\\&\quad=x^{-2}-5x^{-1}-14.\end{aligned}$$

It is seen that the quantities with the negative exponents occupy less space and are quite as easily written.

$$\begin{aligned}&\left(\frac{1}{x}+2\right)\left(\frac{1}{x}-7\right)\\&\quad=\frac{1}{x^2}+\frac{2}{x}-\frac{7}{x}-14\\&\quad=\frac{1}{x^2}-\frac{5}{x}-14.\end{aligned}$$

2. Divide $x^{-3}+1$ by $x^{-1}+1$.

$$\begin{array}{r|l}x^{-3}+1 & x^{-1}+1\\ \hline x^{-3}+x^{-2} & x^{-2}-x^{-1}+1\\ \hline -x^{-2}+1 & \\ -x^{-2}-x^{-1} & \\ \hline x^{-1}+1 & \\ x^{-1}+1 & \\ \hline\end{array}$$

This work is more compact in form than the other and is more easily written. After a little experience it is as easily understood as the other.

$$\begin{array}{r|l}\dfrac{1}{x^3}+1 & \dfrac{1}{x}+1\\ \hline \dfrac{1}{x^3}+\dfrac{1}{x^2} & \dfrac{1}{x^2}-\dfrac{1}{x}+1\\ \hline -\dfrac{1}{x^2}+1 & \\ -\dfrac{1}{x^2}-\dfrac{1}{x} & \\ \hline \dfrac{1}{x}+1 & \\ \dfrac{1}{x}+1 & \\ \hline\end{array}$$

3. Factor $x^{-2}-x^{-1}-56$.

Since the two numbers whose product is -56 and whose sum is -1 are -8 and $+7$, therefore

$$x^{-2}-x^{-1}-56=(x^{-1}-8)(x^{-1}+7).$$

This is only another way of writing

$$\frac{1}{x^2}-\frac{1}{x}-56=\left(\frac{1}{x}-8\right)\left(\frac{1}{x}+7\right).$$

It is therefore an extension of ordinary factoring to include fractions.

Exercise 153. Negative Exponents

Examples 1 to 6, oral — Examples 7 to 33, written

1. What is the value of $4^{\frac{1}{2}}$? of 4^{-2}? of 4^0?
2. What is the value of $9^{\frac{1}{2}}$? of $9^{-\frac{1}{2}}$? of 9^0?
3. What is the value of $8^{\frac{1}{3}}$? of $8^{\frac{2}{3}}$? of $8^{-\frac{2}{3}}$? of 8^0?
4. What is the value of $16^{\frac{1}{4}}$? of $16^{\frac{3}{4}}$? of $16^{-\frac{3}{4}}$? of 16^0?
5. What is the value of 1^{-1}? of 1^{-2}? of 1^{-3}? of $1^{\frac{1}{2}}$? of 1^0?
6. What is the value of 2^{-1}? of 2^{-2}? of 2^{-3}? of 2^0?

Multiply:

7. $(x^{-1}+7)(x^{-2}-3)$.
8. $(x^{-3}+2)(x^{-3}-2)$.
9. $(x^{-2}-2x^{-1}+1)(x^{-1}-1)$.
10. $(x^{-1}+3)(x^{-2}+x^{-1}-2)$.

Divide:

11. $x^{-2}+2x^{-1}+1$ by $x^{-1}+1$.
12. $x^{-3}-3x^{-2}y^{-1}+3x^{-1}y^{-2}-y^{-3}$ by $x^{-1}-y^{-1}$.
13. $x^{-4}-16y^{-4}$ by $x^{-2}+4y^{-2}$; by $x^{-1}+2y^{-1}$.

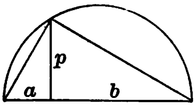
Factor:

14. $x^{-2}-8x^{-1}+7$.
15. $x^{-2}+6x^{-1}-7$.
16. $2a^{-2}+8a^{-1}+8$.
17. $2a^{-2}+7a^{-1}+6$.

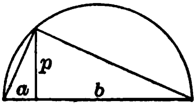
Express in integral form with negative exponents:

18. $\dfrac{1}{x}$.
19. $\dfrac{2}{x^4}$.
20. $\dfrac{3}{x^2y^2}$.
21. $\dfrac{5}{x^2y^3}$.
22. $\dfrac{1}{x^{\frac{1}{2}}}$.
23. $\dfrac{1}{a^{\frac{2}{3}}}$.
24. $\dfrac{2}{a^{\frac{3}{4}}}$.
25. $\dfrac{5}{a^{\frac{1}{2}}}$.
26. $\dfrac{x^2y^2}{x^3y^3}$.
27. $\dfrac{x^2y^3}{x^3y^2}$.
28. $\dfrac{\sqrt{xy}}{xy}$.
29. $\dfrac{a\sqrt{b}}{b\sqrt{a}}$.
30. $\dfrac{2}{a^3}-\dfrac{3}{a^2}+\dfrac{1}{a}$.
31. $\dfrac{4}{a^3}+\dfrac{3}{a^2}-\dfrac{1}{a}$.
32. $\dfrac{x^{\frac{1}{2}}}{x}-\dfrac{y^{\frac{1}{3}}}{y}$.
33. $\dfrac{x^{\frac{2}{3}}}{x}-\dfrac{y^{\frac{3}{4}}}{y}$.

254. Graphic Representation of Surds. It is shown in geometry that if, in a semicircle, a perpendicular is drawn from any point on the diameter and extended to meet the circumference, the perpendicular is a mean proportional between the two parts into which it divides the diameter.



In the figure, $a:p = p:b$, and therefore (§ 163, 1) $p^2 = ab$, or $p = \sqrt{ab}$. If we wish to draw a line equal to $\sqrt{3}$, we make $a = 1$, taking any convenient unit, $\frac{1}{4}$ inch being taken here. We then take $b = 3$ and describe the semicircle. Draw the perpendicular p. Then $p = \sqrt{1 \cdot 3} = \sqrt{3}$. By measuring p the length will be found to be about 1.7, the exact length being $\sqrt{3}$, or 1.73 +.



Required to draw a line equal to $\sqrt{5}$.

In the figure, $a = 1$ and $b = 5$. Then $p = \sqrt{1 \cdot 5} = \sqrt{5}$.

Exercise 154. Graphic Representation of Surds

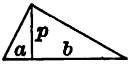
Examples 1 to 7, oral — Examples 8 to 25, written

1. In the above figures, if $a = 1$ and $b = 7$, what does p equal? If $a = 1$ and $b = 9$, what does p equal?

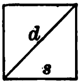
In the above figures find the value of p when:

2. $a = 1, b = 2$.
3. $a = 1, b = 4$.
4. $a = 1, b = 6$.
5. $a = 2, b = 3$.
6. $a = 5, b = 7$.
7. $a = 7, b = 11$.
8. $a = 7, b = 13$.
9. $a = 11, b = 21$.
10. $a = 13, b = 29$.

11. In a right triangle the perpendicular from the vertex of the right angle to the hypotenuse is also a mean proportional between the two parts of the hypotenuse. In the figure, if $a = 1.3$ and $b = 4$, what is the length of p?



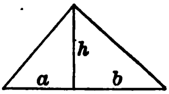
12. In this square, show by § 226 that $d = s\sqrt{2}$.



13. Draw a line $2\sqrt{2}$ inches long, by the method of Ex. 12.

14. In the right triangle below, if $a = 3$ and $b = 4$, what is the value of h? If $a = 4$ and $b = 5$, what is the value of h?

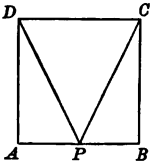
15. In Ex. 14, find the value of h if $a = 5$ and $b = 10$.



16. In Ex. 14, find the value of b if $h = 8$ and $a = 5$.

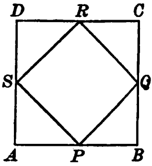
17. In the square below, AP is half of AB. If the side of the square is s, find the length of PC; of $PC + PD$.

In all such cases express the result in the simplest form.



18. In the figure at the right, if $PC = \sqrt{20}$, what is the length of a side of the square?

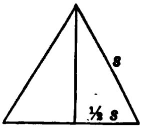
19. In the square below, P, Q, R, and S are midpoints of the sides. If the perimeter of the outer square is 16, what is the perimeter of the inner square?



20. In the figure at the right, if the perimeter of the inner square is 8, what is the perimeter of the outer square?

21. If the side of an equilateral triangle is 2, what is the altitude? If the side is s, what is the altitude?

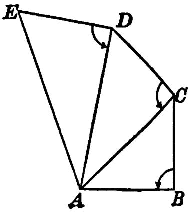
22. If the altitude of an equilateral triangle is h, what is the length of each side? What is the area?



23. If the two sides of a right triangle are $\frac{1}{2}s$ and $\frac{1}{2}s\sqrt{3}$, what is the length of the hypotenuse?

24. If the two sides of a right triangle are x and $\frac{1}{2}x(\sqrt{5} - 1)$, what is the length of the hypotenuse?

25. In the figure at the right, if the angles marked by arrows are all right angles, and if $AB = BC = CD = DE = 1$, show that $AC = \sqrt{2}$, $AD = \sqrt{3}$, and $AE = \sqrt{4} = 2$.



255. Equations involving Radicals. Certain forms of equations containing radicals can be solved by the operations already studied. The following are types:

1. Solve the equation $x = 3 + \sqrt{x^2 - 45}$.

Subtracting 3, $x - 3 = \sqrt{x^2 - 45}$.

Squaring, $x^2 - 6x + 9 = x^2 - 45$.

Subtracting $x^2 + 9$, $-6x = -54$.

Dividing by -6, $x = 9$.

2. Solve the equation $\sqrt{x - 0.1} = \sqrt{x - 3.1} + 1$.

Squaring, $x - 0.1 = x - 3.1 + 2\sqrt{x - 3.1} + 1$.

Subtracting $x - 3.1 + 1$, $2 = 2\sqrt{x - 3.1}$.

Dividing by 2, $1 = \sqrt{x - 3.1}$.

Squaring, $1 = x - 3.1$.

Adding 3.1, $4.1 = x$.

Exercise 155. Equations involving Radicals

Examples 1 to 7, oral — Examples 8 to 19, written

1. Solve $\sqrt{x + 7} = 5$; $\sqrt{x - 7} = 3$; $\sqrt{2x - 5} = 5$.
2. Solve $\sqrt{x - 3} - 4 = 0$; $\sqrt{x - 9} - 5 = 0$; $\sqrt{x} - 2 = 7$.
3. Solve $\sqrt{x}(\sqrt{2} + 1) = \sqrt{2} + 1$; $\sqrt{2x} + \sqrt{x} = \sqrt{2} + 1$

Solve the following equations:

4. $5 + 2\sqrt{x} = 13$.
5. $7 + 4\sqrt{x} = 11$.
6. $\sqrt{3x} - 3 = 3$.
7. $\sqrt[3]{x - 1} = 3$.
8. $\sqrt{12 + x} = 6 - \sqrt{x}$.
9. $\sqrt{x + 1} = 2 + \sqrt{x - 7}$.
10. $\sqrt{10 - x} = \sqrt{22 - x} - 2$.
11. $\dfrac{5x - 1}{\sqrt{5x} + 1} = \dfrac{\sqrt{5x} + 1}{2}$.
12. $\sqrt{4x} - \sqrt{x} = 2$.
13. $\sqrt{9x} - \sqrt{0.25x} = 8$.
14. $\sqrt{x}(\sqrt{2} + 1) = 1$.
15. $\sqrt{2x} + \sqrt{x} = 1$.
16. $2\sqrt{x} - \sqrt{2x} = 2 + \sqrt{2}$.
17. $3\sqrt{x} + \sqrt{5x} = 3 + \sqrt{5}$.
18. $1 + \sqrt{x} = 2(1 - \sqrt{x})$.
19. $\dfrac{3x - 7}{\sqrt{3x} - \sqrt{7}} = \sqrt{7} + 3$.

256. Simplification of an Imaginary. Because $\sqrt{ab}=\sqrt{a}\cdot\sqrt{b}$, we see that $\sqrt{-a}=\sqrt{a(-1)}=\sqrt{a}\cdot\sqrt{-1}$. Therefore an imaginary expression (§ 209) may be written as the product of a real expression and $\sqrt{-1}$.

This is advisable in most of the operations with imaginaries.

Imaginaries are met in the solution of certain equations, and therefore a brief discussion is needed at this time. A more elaborate treatment of the subject is given in advanced courses in algebra.

257. Powers of $\sqrt{-1}$. Because $\sqrt{-1}$ enters into all imaginary expressions of the form $\sqrt{-a}$, since this equals $\sqrt{a}\cdot\sqrt{-1}$, it is necessary to know the various powers of $\sqrt{-1}$. Since the square of the square root of any number is the number itself, therefore $(\sqrt{-1})^2=-1$. Then we have:

$$(\sqrt{-1})^2=-1;$$
$$(\sqrt{-1})^3=(\sqrt{-1})^2\sqrt{-1}=(-1)\sqrt{-1}=-\sqrt{-1};$$
$$(\sqrt{-1})^4=(\sqrt{-1})^2(\sqrt{-1})^2=(-1)(-1)=+1;$$
$$(\sqrt{-1})^5=(\sqrt{-1})^4\sqrt{-1}=(+1)\sqrt{-1}=+\sqrt{-1},$$

and so on, repeating the group $\sqrt{-1}$, -1, $-\sqrt{-1}$, and $+1$.

Frequently i is used for $\sqrt{-1}$, it being the initial of *imaginary*. In that case $i^2=-1$, $i^3=-i$, $i^4=1$, $i^5=i$, $i^6=-1$, $i^7=-i$, $i^8=1$, and so on, the even powers being real and the odd powers imaginary.

258. Product of Two Imaginaries. If the imaginary number $\sqrt{-a}$ is written as above explained, with $\sqrt{-1}$ by itself, the product of such numbers is easily found, thus:

$$\begin{aligned}\sqrt{-a}\cdot\sqrt{-b}&=\sqrt{a}\cdot\sqrt{-1}\cdot\sqrt{b}\cdot\sqrt{-1}\\&=\sqrt{a}\cdot\sqrt{b}\cdot(\sqrt{-1})^2=\sqrt{a}\cdot\sqrt{b}\cdot(-1)\\&=-\sqrt{ab}.\end{aligned}$$

That is, $$\boldsymbol{\sqrt{-a}\cdot\sqrt{-b}=-\sqrt{ab}.}$$

Similarly,
$$\begin{aligned}(-\sqrt{-2})(-\sqrt{-3})&=(-\sqrt{2})\sqrt{-1}(-\sqrt{3})\sqrt{-1}\\&=(+\sqrt{6})(\sqrt{-1})^2\\&=-\sqrt{6}.\end{aligned}$$

259. Complex Number. The sum of a real number and an imaginary number is called a *complex number.*

Thus $2+\sqrt{-3}$, or $2+\sqrt{3}\sqrt{-1}$ is a complex number. In the study of certain equations we need to be able to add and multiply complex numbers.

The general form of a complex number such as we shall meet in our work is $a+b\sqrt{-1}$. Thus $5-\sqrt{-20}=5-\sqrt{4}\sqrt{5}\sqrt{-1}=5-2\sqrt{5}\sqrt{-1}$. Here $a=5$, and $b=-2\sqrt{5}$.

260. Sum of Two Complex Numbers. Complex numbers are added as follows:

$$\begin{array}{r} 3+4\sqrt{-1} \\ 5-7\sqrt{-1} \\ \hline 8-3\sqrt{-1} \end{array} \qquad\qquad \begin{array}{r} a+b\sqrt{-1} \\ p+q\sqrt{-1} \\ \hline (a+p)+(b+q)\sqrt{-1} \end{array}$$

Similarly,
$$\begin{array}{rl} 3+\sqrt{-8}= & 3+\quad 2\sqrt{2}\sqrt{-1} \\ -2+3\sqrt{-16}= & -2+\quad 12\sqrt{-1} \\ \cline{2-2} & 1+(12+2\sqrt{2})\sqrt{-1} \end{array}$$

261. Product of Two Complex Numbers. Complex numbers are multiplied (§ 258) as follows:

$$\begin{array}{r} 2+\ 3\sqrt{-1} \\ 5-\ 2\sqrt{-1} \\ \hline 10+15\sqrt{-1} \\ -\ 4\sqrt{-1}+6 \\ \hline 10+11\sqrt{-1}+6 \\ =16+11\sqrt{-1} \end{array} \qquad\qquad \begin{array}{r} a+\ b\sqrt{-1} \\ p+\ q\sqrt{-1} \\ \hline ap+pb\sqrt{-1} \\ aq\sqrt{-1}-bq \\ \hline ap+(pb+aq)\sqrt{-1}-bq \\ =(ap-bq)+(pb+aq)\sqrt{-1} \end{array}$$

262. Conjugate Complex Numbers. Two numbers of the forms $a+b\sqrt{-1}$ and $a-b\sqrt{-1}$ are called *conjugate complex numbers.*

Since $\quad (a+b\sqrt{-1})+(a-b\sqrt{-1})=2a,$

and
$$\begin{aligned} &(a+b\sqrt{-1})(a-b\sqrt{-1}) \\ &\quad = a^2-ab\sqrt{-1}+ab\sqrt{-1}+b^2 \\ &\quad = a^2+b^2, \end{aligned}$$

the sum or the product of two conjugate complex numbers is a real number.

Exercise 156. Imaginary and Complex Numbers

Examples 1 to 8, oral — Examples 9 to 35, written

1. Express in the form $a\sqrt{-1}$: $\sqrt{-4}$; $\sqrt{-9}$; $\sqrt{-16}$.

2. Express in the form $a\sqrt{-1}$: $\sqrt{-49}$; $\sqrt{-81}$; $\sqrt{-121}$.

Express in the form $a + b\sqrt{-1}$:

3. $2 + \sqrt{-25}$.
4. $3 - \sqrt{-64}$.
5. $7 + \sqrt{-100}$.
6. $3 - \sqrt{-400}$.
7. $2 + \sqrt{-3}$.
8. $3 - \sqrt{-7}$.

Reduce to the form $a\sqrt{-1}$:

9. $2\sqrt{-289}$.
10. $4\sqrt{-361}$.
11. $\frac{1}{2}\sqrt{-441}$.
12. $\frac{1}{4}\sqrt{-784}$.
13. $\frac{1}{9}\sqrt{-81x^2}$.
14. $\sqrt{-a^2b^4c^6}$.
15. $\sqrt{-\frac{1}{9}}$.
16. $\sqrt{-7a^2b^2}$.
17. $\sqrt{-12x^4y^8}$.

Simplify:

18. $\sqrt{-4} + \sqrt{-9}$.
19. $\sqrt{-36} - \sqrt{-25}$.
20. $\sqrt{-144} + \sqrt{-49}$.
21. $\sqrt{-625} - \sqrt{-361}$.
22. $(2 + 3\sqrt{-1}) + (3 - 7\sqrt{-1}) + (-5 + 2\sqrt{-1})$.
23. $(3 - 2\sqrt{-8}) + (5 + 7\sqrt{-18}) + (7 - 3\sqrt{-50})$.
24. $(a + b\sqrt{-1}) + (b + c\sqrt{-1}) + (c + a\sqrt{-1})$.

25. Show that the sum and the product of $5 + 7\sqrt{-1}$ and $5 - \sqrt{-49}$ are both real numbers.

Multiply:

26. $(2 + \sqrt{-1})(2 - \sqrt{-1})$.
27. $(3 + \sqrt{-1})(2 - \sqrt{-1})$.
28. $(4 + 3\sqrt{-1})(2 + \sqrt{-1})$.
29. $(3 + 4\sqrt{-1})(2 - \sqrt{-9})$.
30. $(2 + 3\sqrt{-5})(3 - 2\sqrt{-5})$.
31. $(4 + 7\sqrt{-5})(3 + 6\sqrt{-5})$.
32. $(2\sqrt{3} + 5\sqrt{-3})(3\sqrt{3} + 2\sqrt{-3})$.
33. $(3\sqrt{5} + 2\sqrt{-7})(5\sqrt{5} - 4\sqrt{-7})$.

34. Does $x = 2 \pm \sqrt{-1}$ satisfy the equation $x^2 - 4x + 5 = 0$?

35. Does $x = -\frac{1}{2} - \frac{1}{2}\sqrt{-3}$ satisfy the equation $x^2 + x + 1 = 0$?

CHAPTER XVII

QUADRATIC EQUATIONS

263. Quadratic Equation. An equation which, when reduced to its simplest form, contains the second power, but no higher power, of an unknown quantity is called a *quadratic equation.*

For example, $x^2 = 9$, $x^2 - 3x = 0$, and $x^2 - 3x + 7 = 0$ are quadratic equations in x.

A quadratic equation may always be reduced to the type $ax^2 + bx + c = 0$, in which a is not zero but b or c, or both b and c, may be zero.

Thus $3x^2 + 2x = 7$ is a quadratic equation in which $a = 3$, $b = 2$, and $c = -7$; $5x^2 - 2 = 0$ is a quadratic equation in which $a = 5$, $b = 0$, and $c = -2$. In $2x^2 - 3x = 0$, $a = 2$, $b = -3$, and $c = 0$; and in $4x^2 = 0$, $a = 4$, $b = 0$, and $c = 0$.

264. Coefficients of an Equation. In the equation $ax^2 + bx + c = 0$, a, b, and c are called the *coefficients of the equation.*

If a, b, and c are numbers expressed by figures, the equation is called a *numerical quadratic;* if some or all are represented by letters, the equation is called a *literal quadratic.*

Thus $x^2 + 7x - 3 = 0$ is a numerical quadratic, and $x^2 + ax + m = 5$ is a literal quadratic.

The term represented by c is called the *absolute term*, or the *constant term*, of the equation.

265. Affected Quadratic. A quadratic equation that contains both the second and first powers of the unknown quantity is called an *affected quadratic.*

Thus $x^2 - 5x + 6 = 0$ is an affected quadratic.

An affected quadratic is also called a *complete quadratic.*

266. Pure Quadratic. If the first power of the unknown quantity is missing, the equation is called a *pure quadratic.*

Thus $x^2 - 4 = 0$ is a pure quadratic.

A pure quadratic is also called an *incomplete quadratic.*

1. Solve the equation $5x^2 + 15 = 3x^2 + 65$.

Given $5x^2 + 15 = 3x^2 + 65$.

Subtracting 15 and $3x^2$, $2x^2 = 50$.

Dividing by 2, $x^2 = 25$.

Extracting the square root, $x = \pm 5$.

Check. Substitute either $+5$ or -5 for x in the *given* equation.

Then $5 \cdot 25 + 15 = 3 \cdot 25 + 65$,

or $140 = 140$.

When we extracted the square root of x^2 and 25 we might have written the result $\pm x = \pm 5$. In this case we should then have

$$+x = +5, \quad +x = -5, \quad -x = +5, \quad -x = -5.$$

But since we wish only the values of $+x$, we should obtain from $-x = +5$ the value $x = -5$, and from $-x = -5$ the value $x = 5$, which we have already. Therefore *it is unnecessary to write the sign ± before both members.*

There are therefore only two roots. In this case these are real, rational, and numerically equal, but of opposite signs.

2. Solve the equation $x^2 + 7 = 14$.

Given $x^2 + 7 = 14$.

Subtracting 7, $x^2 = 7$.

Extracting the square root, $x = \pm\sqrt{7}$.

Here the two roots are real but irrational. We may extract the square root of 7 to as many decimal places as we please.

Check. $(\pm\sqrt{7})^2 + 7 = 7 + 7 = 14$.

3. Solve the equation $x^2 + 5 = 0$.

Given $x^2 + 5 = 0$.

Subtracting 5, $x^2 = -5$.

Extracting the square root, $x = \pm\sqrt{-5}$.

Here the two roots are imaginary. We may show that they are correct by substituting in the given equation.

4. Solve the equation $x^2 + 9 = 9$.

Given $x^2 + 9 = 9$.

Subtracting 9, $x^2 = 0$.

Extracting the square root, $x = 0$.

Of course we may write ± 0 instead of 0, if we wish, the value being the same. That is, we have two roots, as is always the case in quadratic equations, and these roots in this example are both zero.

Exercise 157. Pure Quadratics

Examples 1 to 5, oral — Examples 6 to 19, written

1. Solve the equation $x^2 = 25$, and check.

2. Solve the equations $x^2 = 9$ and $x^2 - 9 = 0$.

3. Solve the equations $x^2 - 36 = 0$ and $x^2 - 64 = 0$.

Solve the following equations:

4. $x^2 - 81 = 0$.

5. $x^2 - 121 = 0$.

6. $4x^2 - 196 = 0$.

7. $11x^2 - 176 = 0$.

8. $x^2 - a^2 = 0$.

9. $x^2 - a = 0$.

10. $x^2 + a = 0$.

11. $5t^2 + 7 = 3t^2 + 25$.

Solve the following equations, and check the roots:

12. $\dfrac{35 + 3x}{x + 1} = \dfrac{x - 55}{3x - 53}$.

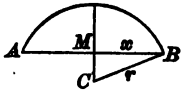
13. $\dfrac{x - 2}{3x + 14} = \dfrac{24 - 3x}{28 - x}$.

14. $\dfrac{x^2 + 5x + 7}{x + 1} = \dfrac{x^2 + x + 1}{x - 1}$.

15. $\dfrac{a + x}{b + x} = \dfrac{x - a}{b - x}$.

16. An arch 6 ft. high is described with a radius 10 ft. Find the length of the span.

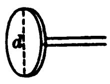
Let $MB = x$. What is the length of MC? Find the value of x; of $2x$.



17. A cistern with a square base is 8 ft. deep. When full it contains 7260 gallons. Allowing $7\frac{1}{2}$ gallons to the cubic foot, find the side of the base.

18. What must be the diameter d of the piston shown in the figure below, to have a steam pressure of 100 lb. per square inch exert a total pressure on the piston of 3850 lb.?

Use $\frac{22}{7}$ for π in this example.



19. Two men start from the same place at the same time. One walks east at the rate of 3 mi. an hour and the other walks north at the rate of 4 mi. an hour. How soon will they be 15 mi. apart?

267. Solving by Factoring. The method of solving an affected quadratic by factoring will be understood from a single example. Required to solve the equation $x^2-5x+6=0$.

Given $x^2-5x+6=0$.

Factoring, $(x-2)(x-3)=0$.

Since the product of $x-2$ and $x-3$ equals 0, one or the other of these factors must equal 0, because no two numbers can have 0 for a product unless one of them is 0. Furthermore, it makes no difference which factor is 0, since 0 multiplied by any other number is 0.

If $x-2=0$,

$x=2$;

and if $x-3=0$,

$x=3$.

Therefore x may equal 2 or it may equal 3. Since there can be only two factors of the polynomial, there can be only two roots.

Check. Substituting 2 in the given equation,

$$2^2-5\cdot2+6=4-10+6=0.$$

Substituting 3, $3^2-5\cdot3+6=9-15+6=0$.

Exercise 158. Solving by Factoring

Examples 1 to 14, oral — Examples 15 to 32, written

1. Factor x^2-6x+8; $x^2-7x+10$; $x^2-8x+15$.
2. Factor $x^2-10x+21$; $x^2-9x+20$; $x^2-10x+24$.
3. Factor $x^2-11x+28$; $x^2-11x+30$; $x^2-12x+35$.
4. Solve the equation $(x-1)(x-2)=0$.
5. Solve the equation $x^2-3x+2=0$.
6. Solve the equation $(x-2)(x+3)=0$.

Solve the following equations:

7. $x(x-2)=0$.
8. $x(x+7)=0$.
9. $x(x-3)=0$.
10. $x^2-3x=0$.
11. $(x-1)(x-5)=0$.
12. $(x-1)(x+5)=0$.
13. $(x+2)(x-9)=0$.
14. $(x-2)(x-9)=0$.

Solve the following equations:

15. $x^2 - 7x + 12 = 0.$

16. $x^2 - 11x + 30 = 0.$

17. $x^2 - 12x + 35 = 0.$

18. $x^2 + 45 = 14x.$

19. $x^2 + 50 = 15x.$

20. $x^2 + 78 = 19x.$

21. $x^2 + 4x - 45 = 0.$

22. $x^2 + 5x - 50 = 0.$

23. $y^2 + 4y = 77.$

24. $u^2 - 7u = 60.$

25. $t^2 = 7t + 78.$

26. $6x^2 + x = 1.$

27. The area of a rectangle is 12 sq. ft. The width is 1 ft. less than the length. Find the dimensions.

Which root is it reasonable to take?

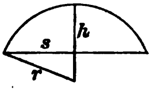
28. The area of a rectangle is 45 sq. in. The length is 4 in. greater than the width. Find the dimensions.

29. A man purchased part of a sheet of 2-cent postage stamps for $1.60. When he was folding it he saw that there were two more rows on one side than on the other. How many stamps were there on each side?

30. If a bar of iron weighing 80 lb. were drawn out 1 ft. longer, it would weigh $6\frac{2}{3}$ lb. less per linear foot. How long is it?

If it is x feet long, what is its present weight per foot? If it were 1 ft. longer, what would be its weight per foot? How do these compare? Which root is it reasonable to take?

31. In building a circular arch the radius is found by the formula $r = \frac{s^2 + h^2}{2h}$, in which $r =$ the radius of the circle, $s =$ half the span of the arch, and $h =$ the height of the arch. If r is known to be 10, and the span ($2s$) to be 16, find the value of h.

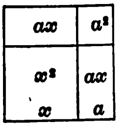


Which root is it reasonable to take?

32. A tank that has two pipes can be filled in 2 hr. less time by one pipe than by the other, and by both together in 2 hr. 55 min. How long will it take each pipe to fill the tank?

268. Completing the Square. A second method of solving the quadratic equation, and one commonly used, is that called the solution by *completing the square.*

Since $(x + a)^2 = x^2 + 2ax + a^2$, the binomial $x^2 + 2ax$ lacks only a^2 of being a perfect square. This a^2 is the square of half the coefficient of x. Therefore if we have an equation in the form $x^2 + 2ax = b$, we may make the first member a perfect square (*complete the square*) by adding a^2 to both members.



For example, given $x^2 + 6x = 55.$

Adding $(\frac{6}{2})^2$, or 3^2, $x^2 + 6x + 9 = 64.$

Extracting the square root, $x + 3 = \pm 8.$

$$\therefore x = -3 \pm 8$$
$$= 5 \text{ or } -11.$$

Check. $5^2 + 6 \cdot 5 = 25 + 30 = 55.$

$$(-11)^2 + 6 \cdot (-11) = 121 - 66 = 55.$$

If the equation is in the form $x^2 - 2ax = b$, we may evidently complete the square in the same way, by adding a^2, the square of $-a$, to both members. We then have

$$x^2 - 2ax + a^2 = b + a^2;$$

whence $x - a = \pm\sqrt{b + a^2},$

and $x = a \pm \sqrt{b + a^2}.$

If the equation has a coefficient for x^2 other than 1, we may divide both members by this coefficient and reduce the equation to the form $x^2 + 2ax = b$.

For example, given $2x^2 - 3x = 9.$

Dividing by 2, $x^2 - \frac{3}{2}x = \frac{9}{2},$

or $x^2 - 2 \cdot \frac{3}{4}x = \frac{9}{2},$

in which $-\frac{3}{4}$ equals the a of the equation $x^2 + 2ax = b$.

Therefore to solve an affected quadratic:

1. *Reduce the equation to the form* $x^2 + 2ax = b$
2. *Add to each member the square of half the coefficient of* x.
3. *Extract the square root of each member of the resulting equation.*
4. *Solve the two resulting simple equations.*

1. Solve the equation $2x^2 - 3x = 9$.

Given $2x^2 - 3x = 9$.

Dividing by 2, $x^2 - \frac{3}{2}x = \frac{9}{2}$.

Adding the square of $\frac{1}{2} \cdot \frac{3}{2}$; that is, adding $(\frac{3}{4})^2$, or $\frac{9}{16}$,

$$x^2 - \tfrac{3}{2}x + \tfrac{9}{16} = \tfrac{9}{2} + \tfrac{9}{16}$$
$$= \tfrac{81}{16}.$$

Extracting the square root, $x - \frac{3}{4} = \pm \frac{9}{4}$.

Solving for x, $x = \frac{3}{4} \pm \frac{9}{4}$

$$= 3 \text{ or } -\tfrac{3}{2}.$$

Check. $2 \cdot 3^2 - 3 \cdot 3 = 18 - 9 = 9$.

$$2 \cdot (-\tfrac{3}{2})^2 - 3(-\tfrac{3}{2}) = \tfrac{9}{2} + \tfrac{9}{2} = 9.$$

In solving this and similar equations we may multiply both members by 2 and have

$$(2x)^2 - 3(2x) = 18;$$

whence $(2x)^2 - 3(2x) + \frac{9}{4} = \frac{81}{4}$, and $2x - \frac{3}{2} = \pm \frac{9}{2}$; whence $x = 3$ or $-\frac{3}{2}$ as before.

2. Solve the equation $x + \dfrac{2}{x} = 4$.

Given $x + \dfrac{2}{x} = 4$.

Multiplying by x, $x^2 + 2 = 4x$.

Subtracting 2 and $4x$, $x^2 - 4x = -2$.

Adding $(\frac{4}{2})^2$, or 2^2, or 4, $x^2 - 4x + 4 = 2$.

Extracting the square root, $x - 2 = \pm\sqrt{2}$.

Solving for x, $x = 2 \pm \sqrt{2}$.

Check. Substituting $2 + \sqrt{2}$ for x in the original equation,

$$2 + \sqrt{2} + \frac{2}{2 + \sqrt{2}} = 4.$$

Rationalizing the denominator of the fraction,

$$2 + \sqrt{2} + \frac{2(2 - \sqrt{2})}{4 - 2} = 4,$$

or

$$2 + \sqrt{2} + 2 - \sqrt{2} = 4.$$

In the same way we may check for $2 - \sqrt{2}$.

If the practical problem in which such an equation enters requires the square root extracted to two or more decimal places, this should be done. The results would then be $2 \pm 1.414+$, or $3.414+$ and $0.586-$.

Exercise 159. Affected Quadratics

Examples 1 to 8, oral — Examples 9 to 40, written

1. What must be added to $x^2 + 8x$ to complete the square?

2. What must be added to $x^2 + 6x$ to complete the square?

Complete the squares in the following cases:

3. $x^2 - 4x$.

4. $x^2 - 6x$.

5. $x^2 + x$.

6. $x^2 - x$.

7. $x^2 + 5x$.

8. $x^2 - 7x$.

Solve the following equations:

9. $x^2 - 20x - 9 = 60$.

10. $x^2 - 20x - 6 = 70$.

11. $x^2 - 12x - 2 = 26$.

12. $x^2 + 8x - 9 = 200$.

13. $x^2 - 4x - 7 = 110$.

14. $x^2 + 5x - 4 = 100$.

15. $x^2 - 8x - 5 = 100$.

16. $x^2 + x - 82 = 100$.

17. $x^2 + 35x + 300 = 0$.

18. $x^2 + x + 1 = 0$.

19. $x^2 - x + 1 = 0$.

20. $x^2 + 3x + 2 = 0$.

21. $x^2 - 7x + 5 = 0$.

22. $x^2 - 9x + 7 = 0$.

23. $2x^2 - 8x - 3 = 0$.

24. $3x^2 - 6x + 10 = 0$.

25. $5x^2 - 2x + 30 = 0$.

26. $7x^2 - 28x + 3\frac{1}{2} = 0$.

27. Find the value of p in the equation $p^2 - 7p = 6$.

28. Find the value of t in the equation $2t^2 - 3t = 35$.

29. Find the value of v in the equation $v^2 + 11 = 8v$.

30. Find to two decimal places the value of K in the equation $K(14 - K) = 47$.

31. Find to three decimal places the value of P in the equation $P^2 = 6(3P - 13)$.

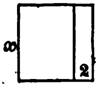
32. Solve the equation $\dfrac{5x - 7}{9} + \dfrac{14}{2x - 3} = x - 1$.

33. Solve the equation $\dfrac{6x + 4}{5} - \dfrac{15 - 2x}{x - 3} = \dfrac{7(x - 1)}{5}$.

34. Solve the equation $\dfrac{x - 5}{x + 3} + \dfrac{x - 8}{x - 3} = \dfrac{3(1 - x)}{x^2 - 9}$.

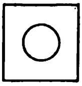
35. If from a square piece of paper I cut a strip 2 in. wide as is shown in the figure, the area of the rest of the paper is 63 sq. in. What is the side of the square?

Let $x =$ the number of inches in the side.
Then $x^2 =$ the area of the square in square inches.
Then $x^2 - 2x = 63.$
Solving, $x = 9$ or $-7.$



Of these roots -7 is inadmissible by the conditions of the problem, although it satisfies the algebraic equation. Usually only one of the algebraic roots meets the conditions of the problem.

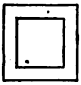
36. A circle with radius 7 in. is cut from a square as shown in the figure. The number of square units of area remaining equals six more than twelve times the side of the square. Find the side of the square. (Use $\frac{22}{7}$ for π.)



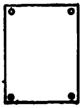
37. A certain positive number multiplied by the sum of itself and 9 equals 220. What is the number? If the problem specified a certain negative number, what would be the result?

38. If twice a certain integer should be subtracted from 7, and the remainder multiplied by the integer, the product would be 6. Find the integer. If the problem specified a fraction instead of an integer, what would be the result? If the problem placed no limitation on the kind of number, what would be the result?

39. The molding for a square picture frame is 2 in. wide. The area of the opening for the picture is 54 sq. in. Find the outside dimensions of the frame. Are both results admissible?



40. A steel plate is 3 in. longer than it is wide. In the four corners holes are drilled for rivets, the area of each opening being 1 sq. in. The area of the rest of the rectangle is 454 sq. in. Find the dimensions of the rectangle. Are both results admissible?



269. Equations solved like Quadratics. It often happens that an equation is not a quadratic in x, but admits of solution by the aid of quadratics.

1. Solve the equation $\frac{(x+2)(x-2)}{5} = \left(\frac{3}{x}\right)^2$.

Clearing of fractions, $x^2(x+2)(x-2) = 45$.

Expanding, $x^4 - 4x^2 = 45$.

Completing the square, $x^4 - 4x^2 + 4 = 49$.

Extracting the square root, $x^2 - 2 = \pm 7$.

Solving, $x = \pm 3$ or $\pm\sqrt{-5}$.

There are therefore four roots, two being real and two being imaginary. *An equation of the fourth degree always has four roots.*

2. Solve the equation $x^3 - 1 = 0$.

Factoring, $(x-1)(x^2+x+1) = 0$.

Then the equation is satisfied if either

$$x - 1 = 0,$$

or if $x^2 + x + 1 = 0$ (by § 267).

If $x - 1 = 0$,

$$x = 1.$$

If $x^2 + x + 1 = 0$,

subtracting 1, $x^2 + x = -1$.

Adding $(\frac{1}{2})^2$, $x^2 + x + \frac{1}{4} = -\frac{3}{4}$.

Solving, $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$.

We therefore have three roots, $1, -\frac{1}{2} + \frac{1}{2}\sqrt{-3}$, and $-\frac{1}{2} - \frac{1}{2}\sqrt{-3}$. *An equation of the third degree always has three roots.*

3. Solve the equation $77x = x^2(18 - x)$.

Rearranging, $x^3 - 18x^2 + 77x = 0$.

Factoring, $x(x^2 - 18x + 77) = 0$.

Then the equation is satisfied if either $x = 0$, or $x^2 - 18x + 77 = 0$.

Then $x^2 - 18x + 77 = 0$,

and $(x-11)(x-7) = 0$.

Solving, $x = 11$ or 7.

Therefore there are three roots, $x = 0$, 11, or 7. *If x is a factor of every term in an equation, 0 is always one of the roots.*

Exercise 160. Equations solved like Quadratics

Examples 1 to 4, oral — Examples 5 to 23, written

1. If $x^4 = 16$, what are the two values of x^2? From these find the four values of x.

2. If $x^4 + 2x^2 = 3$, what must be added to complete the square? Complete the square and find the values of $x^2 + 1$.

3. What are the two roots of the equation $x^2 - x = 0$?

4. What is one of the roots of the equation $x^3 - x = 0$?

Solve the following equations:

5. $x^4 + 8x^2 = 48$.

6. $y^4 - 6y^2 = 27$.

7. $t^4 - 10t^2 = 31$.

8. $p^4 + 10p^2 = 11$.

9. $q^4 + 7q^2 = 30$.

10. $2m^4 + 3m^2 = 119$.

11. $3(n^4 - 21) = 20n^2$.

12. $4n^2(8n^2 + 3) - 5 = 0$.

13. $x^3 + 1 = 0$.

14. $x^3 - 8 = 0$.

15. $x^3 + 27 = 0$.

16. $x^3 + x = 0$.

17. $8x^3 + 1 = 0$.

18. $K^2(K^2 + 1) = 650$.

19. $T^2(T^2 + 9) = 52$.

20. $2P^4 = 3(45 + P^2)$.

21. If the square of a certain number is multiplied by the square increased by 2, the product is 9. What is the number? Is there any integer that satisfies the conditions? Are there any real numbers? any imaginaries?

22. If the square of a certain number is multiplied by the square increased by 3, the product is 28. What is the number? Check the result. Is the number integral? fractional? real? surd? imaginary?

23. In the equation $x^3 = 1$, x must equal the cube root of 1. This equation, in the form of $x^3 - 1 = 0$, is solved on page 318. It therefore appears that 1 has three cube roots. Show that this statement is true by cubing each of the three roots, 1, $-\frac{1}{2} + \frac{1}{2}\sqrt{-3}$, and $-\frac{1}{2} - \frac{1}{2}\sqrt{-3}$.

270. Literal Quadratics. A quadratic with literal coefficients may be solved by completing the square, as shown in the following examples:

1. Solve the equation $x^2 + bx = c$.

Given $x^2 + bx = c$.

Adding $\left(\frac{b}{2}\right)^2$, $x^2 + bx + \frac{b^2}{4} = c + \frac{b^2}{4}$.

Extracting the square root, $x + \frac{b}{2} = \pm\sqrt{c + \frac{b^2}{4}}$

$= \pm \frac{1}{2}\sqrt{4c + b^2}$.

Subtracting $\frac{b}{2}$, $x = -\frac{b}{2} \pm \frac{1}{2}\sqrt{4c + b^2}$.

The amount of checking to be required in the case of such complicated results must be determined by the teacher according to the needs of the class.

2. Solve the equation $(x + a)^2 + 4(x + a) = 21$.

Consider this as a quadratic in $x + a$, and add 4.

Then $(x + a)^2 + 4(x + a) + 4 = 25$.

Extracting the square root, $x + a + 2 = \pm 5$.

Therefore $x = -a + 3 \text{ or } -a - 7$.

Exercise 161. Literal Quadratics

Examples 1 to 3, oral — Examples 4 to 53, written

1. Solve the equations $x^2 = a$; $ax^2 = b$; $a^2x^2 = b^2$.
2. Solve the equations $x(x - a) = 0$; $x(x^2 - a^2) = 0$.
3. Solve the equations $2x(x - 1) = 0$; $ax(x - 1) = 0$.

Solve the following equations:

4. $x^2 + px = 6p^2$.
5. $x^2 - px = 24p^2$.
6. $x^2 + 5ax = 6a^2$.
7. $x^2 + b^2 = ax$.
8. $x^2 - ax = b$.
9. $x^2 + 3cx - 10c^2 = 0$.
10. $x^2 = 5m(10m - x)$.
11. $2x^2 + 3kx = 119k^2$.
12. $x^2 + mx = m^2n(n + 1)$.
13. $x^2 = abc(2abc - x)$.

Solve the following equations:

14. $x^2 = (a - b)x + ab.$

15. $mx^2 = p + 2nx.$

16. $ax^2 + b = 2ax + b^2.$

17. $ax^2 + 5 = 4ax + 7.$

18. $px^2 + 2px = q + r.$

19. $y^2 + 4aby = 5a^2b^2.$

20. $u^2 + 6mu = 16m^2.$

21. $w^2 - 8cw = -16c^2.$

22. $at^2 - a^2bt = 0.$

23. $t^2 - at = a^2k(k - 1).$

24. $v^2 - bv = 1 + b.$

25. $2v^2 + 2bv = 4 - b^2.$

26. $x^2 = k^2(2k^2 - x).$

27. $w(w + 2b) = 24b^2.$

28. $w^2 - (a - b)w = 2b(a + b).$

29. $v(v + al) = l(a + 1).$

30. $4v(v + 4) = l(l + 8).$

31. $x^2 + (a + b)x = 2a(a + b).$

32. $9x^2 + 27ax - 10a^2 = 0.$

33. $a^2x^2 + a^3x = 9 + 3a^2.$

34. $6x^2 - a^2 + 5ax = 0.$

35. $x^2 + x\sqrt{a} = 6a.$

36. $x^2 + 2x\sqrt{ab} = 15ab.$

37. $4x(x + 2\sqrt{-a}) + 5a = 0.$

38. Solve the equation $(a + x)^2 + 2(a + x) = 35$, and check the roots. What are the values of the roots if $a = 1$? if $a = 2$?

Solve the following equations:

39. $\frac{a}{x} + \frac{x}{a} = 1.$

40. $\frac{a}{2x} + \frac{x}{2a} = -1.$

41. $x + \frac{a}{x} = b.$

42. $\frac{x}{a} + a = \frac{1}{x}.$

43. $x - \frac{m}{x} = n.$

44. $x + \frac{1}{x} + a = b.$

45. $x - \frac{1}{x} - a = b.$

46. $x + \frac{a}{x} + b = a.$

47. $x - \frac{a}{x} - b = a.$

48. $ax + \frac{b}{x} + c = d.$

49. $x^2 - (m + n)x + (m + p)(n - p) = 0.$

50. $x^2 - (a - b)x - (a - 1)(b - 1) = 0.$

51. $x^2 + 2ab(a^2 + b^2) = (a + b)^2x.$

52. $(a + b + c)x^2 - (2a + b + c)x + a = 0.$

53. $(a - x)^2 + (b - x)^2 = 2.5(a - x)(b - x).$

271. Solution by Formula. Although the methods of factoring and of completing the square are sufficient for solving any quadratic equation, it is convenient to be able to write down the roots at once without the trouble of factoring or completing the square.

Since 3 × 7 = 7 + 7 + 7, we could dispense with the multiplication table and perform the multiplications by addition. In the same way we could dispense with the solution of the quadratic by formula and continue to solve by the methods already studied. This is not advisable, however, because it usually takes too much time.

Every quadratic equation may evidently be reduced to the form $ax^2 + bx + c = 0$, in which a, b, and c are integers.

Given $$ax^2 + bx + c = 0.$$

Then $$ax^2 + bx = -c.$$

Dividing by a, $$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Completing the square, $$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}.$$

Extracting the square root, $$x + \frac{b}{2a} = \pm\frac{1}{2a}\sqrt{b^2 - 4ac}.$$

Therefore $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Therefore the roots of an equation in the form $ax^2 + bx + c = 0$ *are*

$$\boldsymbol{\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}.$$

If $a = 1$, then $$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

Solve the equation $2x^2 - 3x - 9 = 0$.

Here $a = 2$, $b = -3$, $c = -9$.

Substituting these values in the formula,

$$x = \frac{3 \pm \sqrt{9 + 72}}{4}$$

$$= \frac{3 \pm 9}{4} = 3 \text{ or } -\frac{3}{2}.$$

The student may also solve by factoring or by completing the square.

Exercise 162. Solution by Formula

Examples 1 to 4, oral — Examples 5 to 36, written

1. What is the value of $-\dfrac{b}{2a}$ when $b = -4$, $a = 2$?

2. What is the value of $b^2 - 4ac$ when $b = -4$, $a = 2$, $c = 1$? when $b = 2$, $a = 1$, $c = 1$?

3. When b^2 is greater than $4ac$ are the roots real or are they imaginary?

4. When b^2 is less than $4ac$ are the roots real or are they imaginary?

Solve the following equations, using the formula:

5. $x^2 + 7x + 6 = 0$.

6. $x^2 + 9x + 14 = 0$.

7. $x^2 + x - 12 = 0$.

8. $x^2 - 5x - 6 = 0$.

9. $x^2 + 5x - 6 = 0$.

10. $x^2 - 2x - 15 = 0$.

11. $x^2 - 5x - 14 = 0$.

12. $x^2 - 9x + 14 = 0$.

13. $x^2 + 5x - 14 = 0$.

14. $x^2 - 14x + 33 = 0$.

15. $x^2 - 11x + 28 = 0$.

16. $y^2 - 17y + 72 = 0$.

17. $w^2 - 13w + 42 = 0$.

18. $6x^2 - x - 1 = 0$.

19. $6y^2 - 26y - 1 = 0$.

20. $2t^2 - 5t + 2 = 0$.

21. $5k^2 - 22k + 21 = 0$.

22. $6p^2 + 13p + 6 = 0$.

23. $16m^2 - 49 = 0$.

24. $121x^2 - a^2 = 0$.

25. $a^2x^2 - 4ax - 5 = 0$.

26. $a^2x^2 + 4ax - 21 = 0$.

27. $x^2 - 9x = -9$.

28. $x^2 + 10x = -24$.

29. $x^2 - 2x = 24$.

30. $x^2 + \frac{1}{2}x = \frac{3}{16}$.

31. $a^2x^2 + a(b - c)x - bc = 0$.

32. $x^2 - (a + b)x + ab = 0$.

33. $x^2 + (a - b)x - ab = 0$.

34. $x^2 + (a + 2)x + 2x = 0$.

35. $x^2 + 8x + 49 = 17x + 29$.

36. $3(5x^2 + 12) + 3x = 30 - 13x$.

272. Relation of the Roots to the Coefficients. Designating the two roots of the equation $x^2 + bx + c = 0$ by x_1 and x_2, we have

$$x_1 = \frac{-b + \sqrt{b^2 - 4c}}{2},$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4c}}{2}.$$

Therefore $$\boldsymbol{x_1 + x_2 = -b},$$

and $$\boldsymbol{x_1 x_2 = c}.$$

For $$x_1 + x_2 = \frac{-b + \sqrt{b^2 - 4c}}{2} + \frac{-b - \sqrt{b^2 - 4c}}{2} = \frac{-2b}{2} = -b,$$

and $$x_1 x_2 = \frac{(-b + \sqrt{b^2 - 4c})(-b - \sqrt{b^2 - 4c})}{4}$$

$$= \frac{b^2 - (b^2 - 4c)}{4} = \frac{b^2 - b^2 + 4c}{4} = c.$$

Therefore, in an equation of the form $x^2 + bx + c = 0$,

The product of the roots equals the absolute term.

The sum of the roots equals the coefficient of x with the opposite sign.

Students should be able to apply this check to most equations at sight.

1. Are 3 and − 5 the roots of $x^2 + 2x - 15 = 0$?

We see that $3 \cdot (-5) = -15$, the absolute term;

and $3 + (-5) = -2$, minus the coefficient of x.

Therefore 3 and − 5 are the roots.

2. Are $-\frac{1}{2}$ and -2 the roots of $2x^2 + 3x - 2 = 0$?

Reduce to the form $x^2 + \frac{3}{2}x - 1 = 0$.

Then $(-\frac{1}{2})(-2) = 1$,

which is not the absolute term (− 1). Therefore $-\frac{1}{2}$ and − 2 are not the roots.

3. Form the equation whose roots are 6 and $-\frac{2}{3}$.

In the equation $x^2 + bx + c = 0$ we must have

$$b = -(6 - \tfrac{2}{3}) = -5\tfrac{1}{3} = -\tfrac{16}{3},$$

and $$c = 6 \cdot (-\tfrac{2}{3}) = -4.$$

Therefore the equation is $x^2 - \frac{16}{3}x - 4 = 0$, or $3x^2 - 16x - 12 = 0$.

Exercise 163. Relation of Roots and Coefficients

Examples 1 to 17, oral — Examples 18 to 36, written

1. Are 3 and 6 the roots of $x^2 - 9x + 18 = 0$?

2. Are -7 and -2 the roots of $x^2 - 9x + 14 = 0$?

3. What is the sum of the roots of $x^2 + 3 = 2x$?

4. What is the product of the roots of $x^2 + 5x = 6$?

Without solving, determine whether the roots of the following equations are as stated:

5. $x^2 - 8x + 15 = 0$; 5, 3.
6. $x^2 - 16x + 72 = 0$; 7, 9.
7. $x^2 - 14x + 48 = 0$; 6, 8.
8. $x^2 - 15x + 44 = 0$; 4, 11.
9. $x^2 + x - 12 = 0$; 3, -4.
10. $x^2 + 12x + 35 = 0$; 5, 7.
11. $x^2 + 15x + 56 = 0$; 7, 8.
12. $x^2 - 12x + 32 = 0$; 4, 8.
13. $x^2 - 4x + 2 = 0$; 2, 2.
14. $x^2 - 6x - 9 = 0$; 3, -3.

15. $x^2 + x + 1 = 0$; $-\frac{1}{2} + \sqrt{2}$, $-\frac{1}{2} - \sqrt{2}$.

16. $x^2 + (a + b)x + ab = 0$; $-a$, $-b$.

17. $x^2 - (a + b + c)x + (a + b)c = 0$; $-(a + b)$, c.

Form the equations of which the roots are:

18. 2, 3.
19. -2, 3.
20. 2, -3.
21. -2, -3.
22. 7, -13.
23. -11, -17.
24. -19, 35.
25. -55, -13.
26. $\frac{2}{3}$, 9.
27. $\frac{3}{5}$, 15.
28. $\frac{3}{4}$, $\frac{7}{8}$.
29. $-\frac{1}{8}$, 24.
30. a, $-b$.
31. $\frac{1}{2}a$, b.
32. a^8, a^2.
33. $-a^4$, $-b^4$.

34. If some one at the blackboard should give 1.22 and 3.06 as the roots of the equation $x^2 - 4.28x + 3.7147 = 0$, how could you tell, without solving, that they were wrong?

35. If you find $2\frac{1}{2}$ and 3.1416 to be the roots of the equation $x^2 - 5.6416x + 7.854 = 0$, how will you check them?

36. If x_1 and x_2 are the two roots of the equation $x^2 + bx + c = 0$, find the value of $\frac{1}{x_1} + \frac{1}{x_2}$.

273. Nature of the Roots. We have found (§ 271) that the roots of the general quadratic equation $ax^2 + bx + c = 0$ are

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \frac{-b-\sqrt{b^2-4ac}}{2a}.$$

We see that if $b^2 - 4ac = 0$ the two radicals become zero and the two roots become $-\frac{b}{2a}$ and $-\frac{b}{2a}$; that is, the two roots are real and equal.

For example, in the equation $x^2 - 4x + 4 = 0$ we have $b^2 - 4ac = 16 - 4 \cdot 4 = 0$. Hence we know, before we solve, that the two roots are real and equal.

If $b^2 - 4ac$ is a perfect square, $\sqrt{b^2 - 4ac}$ is rational, and hence both roots are rational.

For example, in the equation $x^2 - 9x + 14 = 0$ we have $b^2 - 4ac = 81 - 56 = 25$. Hence we know, without solving completely, that the two roots are rational but that they are not equal.

If $b^2 - 4ac$ is positive and not a perfect square, $\sqrt{b^2 - 4ac}$ is irrational, and hence both roots are irrational and unequal.

If $b^2 - 4ac$ is negative, $\sqrt{b^2 - 4ac}$ is imaginary, and hence both roots are imaginary.

274. Discriminant. The expression $b^2 - 4ac$ is called the *discriminant* of the equation $ax^2 + bx + c = 0$.

By its use we discriminate as to the nature of the roots, thus:

If $\mathbf{b^2 - 4ac > 0}$, *the roots are real and unequal;*
if $\mathbf{b^2 - 4ac = 0}$, *the roots are real and equal;*
if $\mathbf{b^2 - 4ac < 0}$, *the roots are imaginary.*

1. What is the nature of the roots of $x^2 - 2x + 1 = 0$?

We have $b^2 - 4ac = (-2)^2 - 4 \cdot 1 \cdot 1 = 4 - 4 = 0$. Hence the discriminant equals 0, and the roots are real and equal. Indeed, we know by inspection that the roots are 1 and 1.

2. What is the nature of the roots of $x^2 - x + 1 = 0$?

We have $b^2 - 4ac = (-1)^2 - 4 \cdot 1 \cdot 1 = 1 - 4 = -3$. Hence the discriminant is negative, and the roots are both imaginary. If we solve, we shall find that the roots are $\frac{1}{2}(1 \pm \sqrt{-3})$.

Exercise 164. Nature of the Roots

Examples 1 to 6, oral — Examples 7 to 26, written

1. What is the discriminant of $x^2 + 2x + 1 = 0$?

2. What is the discriminant of $x^2 + 2x + 2 = 0$?

3. If the discriminant of an equation equals $\frac{2}{3}$, what is the nature of the roots?

4. If the discriminant of an equation equals -0.5, what is the nature of the roots?

5. If the discriminant of an equation equals 0.25, what is the nature of the roots?

6. In the equation $ax^2 + bx + c = 0$ what is the nature of the roots when $b^2 = 4ac$?

Without solving, determine the nature of the roots:

7. $x^2 + x + 1 = 0$.

8. $x^2 + 2x + 3 = 0$.

9. $x^2 - x + 10 = 0$.

10. $x^2 - 3x + 1 = 0$.

11. $x^2 + 4x + 4 = 0$.

12. $x^2 - 2x + 4 = 0$.

13. $x^2 - 2x - 4 = 0$.

14. $x^2 + 7x + 12 = 0$.

15. $5x^2 - 3x - 2 = 0$.

16. $4x^2 - 4x - 1 = 0$.

17. $7x^2 + 9x - 10 = 0$.

18. $3x^2 - 7x - 6 = 0$.

19. $2x^2 - 13x + 15 = 0$.

20. $9x^2 + 12x + 4 = 0$.

21. Give to a, b, and c values that will make $b^2 - 4ac = 0$. Write an equation having two real and equal roots.

22. Give to a, b, and c values that will make $b^2 - 4ac > 0$. Write an equation having two real and unequal roots.

23. Write an equation having two unequal surd roots.

24. Write an equation having two imaginary roots.

25. Write an equation that will have $-2 + \sqrt{-2}$ and $-2 - \sqrt{-2}$ for its roots.

26. For what values of a will the roots of $2x^2 + (1 + a)x + 2 = 0$ be equal? real? imaginary?

275. Special Forms. A few other special forms of equations will now be considered.

1. Solve the equation $x^{2n} + 2x^n - 15 = 0$.

This is a quadratic in x^n. Solving for x^n, we have

$$x^n = \frac{-2 \pm \sqrt{4+60}}{2} = \frac{-2 \pm 8}{2} = 3 \text{ or } -5.$$

$$\therefore x = \sqrt[n]{3} \text{ or } \sqrt[n]{-5}.$$

2. Solve the equation $2x^{\frac{2m}{n}} - 17x^{\frac{m}{n}} + 35 = 0$.

Solving for $x^{\frac{m}{n}}$, $x^{\frac{m}{n}} = \frac{17 \pm \sqrt{17^2 - 4 \cdot 2 \cdot 35}}{2 \cdot 2} = \frac{17 \pm \sqrt{289 - 280}}{4}$

$$= \frac{17 \pm 3}{4} = 5 \text{ or } 3\tfrac{1}{2}.$$

Raising each member to the nth power and then extracting the mth root,

$$x = 5^{\frac{n}{m}} \text{ or } (3\tfrac{1}{2})^{\frac{n}{m}}.$$

3. Solve the equation $(x^2 + 2x)^2 - 2(x^2 + 2x) - 3 = 0$.

Solving for $x^2 + 2x$, $x^2 + 2x = \frac{2 \pm \sqrt{4+12}}{2} = \frac{2 \pm 4}{2} = 3 \text{ or } -1.$

We now have $x^2 + 2x = 3$,

and $x^2 + 2x = -1$.

Solving the first, we have $x = -3 \text{ or } 1$.

Solving the second, we have $x = -1 \text{ or } -1$.

Therefore the roots are $-3, 1, -1, -1$.

4. Solve the equation $x - 8\sqrt{x+3} + 18 = 0$.

We may write this $x + 3 - 8\sqrt{x+3} + 15 = 0$.

Here $x + 3$ is the square of $\sqrt{x+3}$. Therefore, solving for $\sqrt{x+3}$, we have

$$\sqrt{x+3} = 3 \text{ or } 5.$$

$$\therefore x + 3 = 9 \text{ or } 25.$$

$$\therefore x = 6 \text{ or } 22.$$

5. Solve the equation $x^{-2} + 3x^{-1} - 4 = 0$.

This is a quadratic in x^{-1}. Solving, $x^{-1} = -4$ or 1. Since $\frac{1}{x} = -4$ or 1, therefore $x = -\frac{1}{4}$ or 1.

Exercise 165. Special Forms of Quadratics

Examples 1 to 3, oral — Examples 4 to 25, written

1. In the equation $x^{2n} + bx^n + c = 0$ what is the sum of the two values of x^n? What is their product?

2. In the equation $x - 2x^{\frac{1}{2}} + 1 = 0$ how is x related to $x^{\frac{1}{2}}$? For what unknown quantity shall you solve the quadratic?

3. In the equation $x^{\frac{m}{n}} - 2x^{\frac{m}{2n}} + 1 = 0$ how is $x^{\frac{m}{n}}$ related to $x^{\frac{m}{2n}}$? For what unknown quantity shall you solve the quadratic?

Solve the following equations:

4. $x - 8\sqrt{x} + 15 = 0.$

5. $x - 10\sqrt{x} + 21 = 0.$

6. $x - 13x^{\frac{1}{2}} + 40 = 0.$

7. $x^3 - 6x^{\frac{3}{2}} + 8 = 0.$

8. $x^8 - 20x^4 + 99 = 0.$

9. $x^p - 15x^{\frac{p}{2}} + 56 = 0.$

10. $x^{2p} - 13x^p + 30 = 0.$

11. $x^{-2} - 12x^{-1} + 27 = 0.$

12. $x^{-2} - 9x^{-1} + 8 = 0.$

13. $9x^{-2} - 21x^{-1} + 12 = 0.$

14. $(x^2 - 3x)^2 + 3(x^2 - 3x) + 2 = 0.$

15. $(x^2 + 7x)^2 + 5(x^2 + 7x) - 84 = 0.$

16. $(x^2 + 5x + 1)^2 + 4(x^2 + 5x + 1) + 5 = 0.$

17. $x - 9 - 9\sqrt{x - 9} + 20 = 0.$

18. $x^2 + 3x + 2 + 11\sqrt{x^2 + 3x + 2} + 30 = 0.$

19. $(3x - 5)^2 - 8(3x - 5) + 7 = 0.$

20. $x^2 + 5 = 8x + 2\sqrt{x^2 - 8x + 40}.$

21. $3\sqrt{3x^2 - 2x + 4} = 3x^2 - 2x - 6.$

22. $39 - 8x + \sqrt{7x^2 + 8x - 19} - 7x^2 = 0.$

23. $2x^2 + 3\sqrt{x^2 - x + 1} = 2x + 3.$

24. $2x^2 - 6x + \sqrt{x^2 - 3x + 6} - 9 = 0.$

25. In the equation $\frac{x^2}{x+1} + \frac{x+1}{x^2} = \frac{5}{2}$ let $y = \frac{x^2}{x+1}$ and solve for y. Then let $\frac{x^2}{x+1}$ equal these values and solve for x.

276. Extraneous Roots. Sometimes in working with equations a root appears that cannot be checked.

For example, if we have the equation $x - 2 = 5$, and we square both members, we have
$$x^2 - 4x + 4 = 25,$$
or
$$x^2 - 4x - 21 = 0.$$
Solving,
$$x = 7 \text{ or } -3.$$

Of these roots, 7 can be checked because we started with it, and $7 - 2 = 5$; but -3 cannot be checked, for $-3 - 2$ does not equal 5.

Therefore -3 is called an *extraneous root.* This is only another way of saying that -3 is not a root of the original equation, although it is a root of some of the derived equations.

If both members of an equation are raised to the same power, an extraneous root is liable to appear.

It is therefore necessary to check with great care the roots of any equation involving radicals.

Solve $\sqrt{x+3} + \sqrt{x+8} = 5\sqrt{x}.$

Squaring, $x + 3 + 2\sqrt{(x+3)(x+8)} + x + 8 = 25x.$

Hence, $2\sqrt{x^2 + 11x + 24} = 23x - 11.$

Squaring, $4x^2 + 44x + 96 = 529x^2 - 506x + 121.$

Hence, $525x^2 - 550x + 25 = 0.$

Dividing by 25, $21x^2 - 22x + 1 = 0.$

Solving, $x = 1 \text{ or } \frac{1}{21}.$

Of these roots 1 will check and $\frac{1}{21}$ will not. Hence $\frac{1}{21}$ is an extraneous root.

It must be remembered that the radical sign indicates only the principal root (§ 215), and hence we cannot say that
$$\sqrt{64} + \sqrt{169} = -8 + 13 = 5.$$

If both members of an equation are multiplied by some function of the unknown quantity, an extraneous root is liable to appear.

For example, if we multiply $x - 5 = 0$ by $x - 2$, we have $(x-5)(x-2) = 0$. The roots of the latter are 5 and 2. Hence an extraneous root, 2, has been introduced.

In general, in clearing of fractions extraneous roots are not introduced, but it is necessary to check to make sure of this fact.

Exercise 166. Radical Equations

Examples 1 to 4, oral — Examples 5 to 24, written

1. Solve the equation $2\sqrt{x}=5$. Check the result.

2. Solve the equation $\sqrt{x+7}=4$. Check the result.

3. Solve the equation $\sqrt{\frac{1}{2}x+3}=5$. Check the result.

4. In the equation $x-3\sqrt{x}+2=0$ what is the sum of the two values of $\sqrt{x}$? What is the product of these values? What is the product of the two values of x?

Solve the following equations:

5. $x+5=5\sqrt{x-1}$.

6. $\sqrt{x+\sqrt{x+4}}=2\sqrt{2}$.

7. $\sqrt{x+\sqrt{x+2}}=2\sqrt{\frac{5}{2}}$.

8. $\sqrt{\sqrt{x-2}+x}=2$.

9. $\sqrt{x+\sqrt{x+3}+1}=0$.

10. $\sqrt{x+\sqrt{x+9}}=\sqrt{-3}$.

11. $\sqrt{x+7+\sqrt{5(x-2)}}=3$.

12. $\sqrt{x+3}+\sqrt{x+8}=5\sqrt{x}$.

13. $2\sqrt{5+2x}-\sqrt{13-6x}=\sqrt{37-6x}$.

14. $\sqrt{2x+1}+\sqrt{x-3}=2\sqrt{x}$.

15. $\sqrt{5x-1}-\sqrt{8-2x}=\sqrt{x-1}$.

16. $\sqrt{a-x}+\sqrt{b-x}=\sqrt{a+b-2x}$.

17. $\sqrt{(a+x)(x+b)}+\sqrt{(a-x)(x-b)}=2\sqrt{ax}$.

18. $2x^2+a\sqrt{b(b+4x)}=a(b+2x)$.

19. $\sqrt{2a-b+2x}-\sqrt{10a-9b-6x}=4\sqrt{a-b}$.

20. $x+\sqrt{a^2+x^2}=\dfrac{5a^2}{\sqrt{a^2+x^2}}$.

21. $\sqrt{\dfrac{x}{a}}+\sqrt{\dfrac{a}{x}}=\sqrt{\dfrac{x}{b}}+\sqrt{\dfrac{b}{x}}$.

22. $\dfrac{\sqrt{1+x}+\sqrt{x-7}}{\sqrt{1+x}-\sqrt{x-7}}=2$.

23. $12\sqrt{\dfrac{x}{2}}+5\sqrt{\dfrac{2}{x}}=26.5$.

24. $\dfrac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}-\dfrac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}=8\sqrt{x^2-1}$.

Exercise 167. Problems involving Quadratics

Examples 1 to 5, oral — Examples 6 to 52, written

1. State an equation showing that the product of two consecutive numbers is 182.

2. State an equation showing that if 3 is taken from a certain number the result equals 28 divided by the number.

3. State an equation showing that the difference between the cubes of two consecutive numbers is 37.

4. State an equation showing that the sum of a number and its reciprocal is $\frac{145}{12}$.

5. State an equation showing that the sum of the squares of two consecutive numbers is 145; that the sum of the squares of two consecutive even numbers is 100; that the sum of the squares of two consecutive odd numbers is 130.

6. Solve Exs. 1–4.

7. Solve the three problems in Ex. 5.

8. Find a number such that its square is four times the product of the number and 12.

9. Find a number such that its square is four times the sum of the number and 8.

10. Find two consecutive numbers the sum of whose squares is 313.

11. Find two consecutive even numbers the sum of whose squares is 884.

12. The sum of two adjacent sides of a rectangle is 23 in. and the area of the rectangle is 120 sq. in. Find the dimensions.

13. The perimeter of a rectangle is 42 in. and the area is 108 sq. in. Find the dimensions.

14. One side of a rectangle is 5 in. shorter than the other, and the area is 176 sq. in. Find the dimensions.

15. One side of a rectangle is $3\frac{1}{2}$ in. longer than the other, and the area is 200 sq. in. Find the dimensions.

A straight line AB, 1 inch long, is divided by the point P so as to satisfy one of the following conditions. Find the length of AP in each case:

16. $\overline{AP}^2 = AB \cdot PB.$

17. $\overline{AP}^2 = 2\,AB \cdot PB.$

18. $2\,\overline{AP}^2 = 3\,AB \cdot PB.$

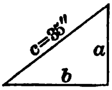
19. $\overline{AP}^2 = 2\,\overline{PB}^2.$

20. $\overline{AP}^2 = \overline{AB}^2 - 2\,\overline{PB}^2.$

21. $\overline{AP}^2 - \overline{PB}^2 = \frac{1}{4}$ sq. in.

22. In the right triangle shown below, the hypotenuse is 35 in. and the side a is 7 in. shorter than the side b. Find the length of each side.

23. In the same figure, if a is 10 in. longer than b, and c is 50 in., what is the length of each side?



24. In the same figure, if a is 33 ft. longer than b, and c is 165 ft., what is the length of each side?

25. A lady planted a square bed of flowers next to her house. Each year the plants spread 3 in. on each of the three open sides. At the end of three years the bed had an area of 4012 sq. in. What was the length of the original bed?

26. A rectangular plot of grass has a perimeter of 232 ft. The length is 4 ft. greater than the width. A walk surrounds the plot, and the area of the walk equals the area of the plot. What is the width of the walk?

27. A cylindric tomato can has its height 1 in. greater than the diameter of the base. To make such a can requires 46 sq. in. of tin, 2 sq. in. of which are lost in overlapping. Taking π as $3\frac{1}{7}$, find the diameter of the base.

28. It is shown in physics that if a body is thrown downward with an initial velocity of V feet per second, the distance d feet that it will pass through in t seconds is given by the formula $d = Vt + 16\,t^2$. If a body is thrown downward from a height of 1900 ft. with an initial velocity of 60 ft. per second, how long will it take it to reach the ground?

29. In Ex. 28, if there is no initial velocity, that is, if the body is simply dropped, the formula becomes $d = 16t^2$. A stone is dropped into a vertical shaft of a mine, and the sound of its striking is heard 10 seconds later. If the velocity of sound is 1120 ft. per second, how deep is the shaft?

30. In Ex. 28, if the body is thrown vertically upward instead of downward, the formula is $d = Vt - 16t^2$. A boy shoots an arrow vertically upward with an initial velocity of 96 ft. per second. How long before it is 80 ft. above the ground? Show that both results are admissible.

31. The perimeter of a right triangle is 108 in. and the hypotenuse is 45 in. Find the length of each side.

32. The area of a right triangle is 384 sq. in. and the hypotenuse is 40 in. Find the length of each side.

33. The hypotenuse of a right triangle is 7 in. longer than one side and 14 in. longer than the other side. Find the length of the hypotenuse and of the two sides.

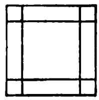
34. A certain number of students pay $20 a week for table board at a boarding house. They find that by preparing their own meals they can buy the food for themselves and one other student for $10 a week, each of the group saving $3 a week on what would be paid at the boarding house. How much does each pay if they prepare their own meals?

If x = the original number of students, $\frac{10}{x+1} + 3 = \frac{20}{x}$.

35. A boy was sent to market with $1.50 to buy some oranges. The price had been raised 10¢ a dozen, so that he bought half a dozen less than he expected to buy. Find the former price, the present price, and the number of oranges purchased.

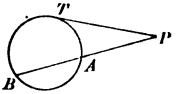
36. A boy rode into the country on his bicycle. After he had gone 10 mi. the chain broke and he was obliged to walk home. He walked 4 mi. less per hour than he rode, and it took him an hour and a quarter longer to return than to go out. How fast did he walk per hour?

37. A square piece of tin is made into a box by cutting from the corners small squares 2 in. on a side. The box then contains 50 cu. in. Required the dimensions of the piece of tin.

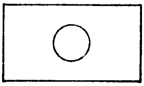


38. A square piece of tin is made into a box by cutting from the corners small squares n inches on a side. The box then contains v cubic inches. Required the dimensions of the piece of tin.

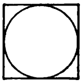
39. It is proved in geometry that $\overline{PT}^2 = PA \cdot PB$, PT being a tangent and PB any line from P meeting the circle twice. If it is known that $PA = 2$, and $PB = 1 + PT$, what is the length of PT?



40. In a steel plate to be used in a steamship is a porthole 14 in. in diameter. The plate is 5 ft. longer than it is wide. The area of the plate, exclusive of the porthole, is $34\frac{67}{72}$ sq. ft. What are the dimensions of the plate? (Take $3\frac{1}{7}$ for π.)



41. The circle shown below is so drawn as just to touch the four sides of the square. The area of that part of the square not covered by the circle is $10\frac{1}{2}$ sq. in. What is the radius of the circle?



42. In a certain trapezoid the upper base is 2 in. and the lower base equals the altitude. The area of the trapezoid is $31\frac{1}{2}$ sq. in. Find the length of the lower base.

43. A broker bought a number of bank shares of the par value of $100 each, when they were at a certain per cent below par, for $7500. Afterwards, when they were at the same per cent above par, he sold all but 60 shares for $5000. How many shares did he buy, and at what price?

44. The thickness of a rectangular solid is $\frac{2}{3}$ of its width, and its length equals the sum of the width and thickness. The number of cubic yards in its volume, added to the number of linear yards in its edges, is $\frac{5}{3}$ of the number of square yards in its surface. Required the dimensions.

45. If the edges of a rectangular box are increased by 2 in., 3 in., and 4 in. respectively, the box becomes a cube and its capacity is increased by 1008 cu. in. Find the dimensions of the box.

46. A railway train makes a run of 799 mi. in a certain time. If the rate of speed should be reduced $4\frac{1}{2}$ mi. an hour, the run would take 108 min. longer. How long does it take the train to make the run?

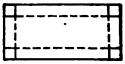
47. A reservoir can be filled by two pipes, A and B, in 9 min., when both are open. The pipe A can fill it in 24 min. less time than B. How long will it take A to fill it?

48. A and B start on two roads that cross at right angles, at distances of 20 and 24 miles respectively from the crossing, and walk toward it at the same rate. How far must they go so that the distance between them is reduced 4 mi.?

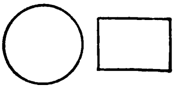
49. A certain rectangle is 1.3 ft. longer than wide. If the length is increased by 0.6 ft. and the width decreased by 0.1 ft., the area is increased by 0.86 sq. ft. Required the original dimensions.

50. A box containing 384 cu. in. is made by cutting out the corners of a square piece of pasteboard and then turning up each side 6 in. What was the area of the original square?

51. A box containing 324 cu. in. is made by cutting out the corners of a rectangular piece of pasteboard that was half as wide as long, and then turning up each side 6 in. What were the dimensions of the rectangle?



52. A string lies on the circumference of a circle and exactly equals it in length. It is then stretched out into a rectangle of which the length is 2 in. more than the width, and of which the area is 120 sq. in. Find the dimensions of the rectangle, the circumference of the circle, the radius of the circle, and the area of the circle. (Take $3\frac{1}{7}$ for π.)



CHAPTER XVIII

SIMULTANEOUS QUADRATIC EQUATIONS

277. Simultaneous General Quadratics. The most general type of quadratic equation with two unknown quantities contains both of these quantities in the first and second degrees, contains their product, has an absolute term, and has coefficients for all the terms containing an unknown quantity.

That is, just as $ax^2 + bx + c = 0$ is the most general quadratic equation with one unknown quantity, so $ax^2 + bxy + cy^2 + dx + ey + f = 0$ is the most general quadratic equation with two unknown quantities.

Two such equations may be represented by

$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

and $$a'x^2 + b'xy + c'y^2 + d'x + e'y + f' = 0.$$

If we eliminated y, we should have an equation of the fourth degree in x that could not be solved by quadratics.

Therefore, *in general, two simultaneous quadratic equations cannot be solved by the method of quadratics.*

There are, however, certain special types that can be solved, and these are considered in this chapter.

That the student may see the difficulties, even in the case of two quadratic equations that seem easy of treatment, consider the system

$$x^2 + y = 7, \qquad (1)$$

$$x + y^2 = 11. \qquad (2)$$

From (1), $$y = 7 - x^2. \qquad (3)$$

Substituting (3) in (2), $$x + (7 - x^2)^2 = 11,$$

whence $$x^4 - 14x^2 + x + 38 = 0.$$

This is an equation of the fourth degree, and although one root ($x = 2$) may easily be found by factoring, it cannot be obtained from this equation by the methods of quadratics.

278. Simple and Quadratic Equations. Before considering two simultaneous quadratic equations we shall consider two simultaneous equations of which one is simple and the other quadratic.

Solve the equations

$$2x^2 + y^2 = 33 \qquad (1)$$

$$2x + y = 9 \qquad (2)$$

Subtracting $2x$ from (2), $y = 9 - 2x$. (3)

Substituting (3) in (1), $2x^2 + (9 - 2x)^2 = 33$,

whence $x^2 - 6x + 8 = 0$.

Solving, $x = 2$ or 4.

Substituting in (3), $y = 5$ or 1.

We therefore have two pairs of roots, in which

$$x = 2 \quad \text{when} \quad y = 5,$$

and

$$x = 4 \quad \text{when} \quad y = 1.$$

For the first pair

equation (1) becomes $2 \cdot 2^2 + 5^2 = 8 + 25 = 33$,

equation (2) becomes $2 \cdot 2 + 5 = 4 + 5 = 9$,

and similarly for the second pair.

In substituting, join with x the corresponding value of y.

That is, $x = 2$ will not check with $y = 1$, nor $x = 4$ with $y = 5$.

Exercise 168. Simple and Quadratic Equations

Examples 1 and 2, oral — Examples 3 to 8, written

1. Solve the equations $x = y$, $x^2 + y^2 = 2$.
2. Solve the equations $x = -y$, $x^2 + y^2 = 2$.

Solve the following equations:

3. $3x^2 + y^2 = 43$
   $5x - y = 11$
4. $2x^2 + y^2 = 51$
   $\frac{1}{5}x = -y$
5. $xy = -15$
   $4x + 7y = 23$
6. $3x - y = 5$
   $xy - x = 0$
7. $x^2 - xy + y^2 = 7$
   $2x - 3y = 0$
8. $x^2 - xy - 2y^2 = 7$
   $x - y - 3 = 0$

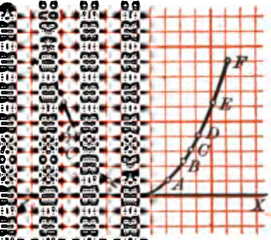
[illegible] stand more [illegible] equation it is [illegible] we already [illegible] ary at this [illegible].

[illegible]

[illegible] We therefore [illegible]

[illegible]	9
[illegible]	± 6
[illegible]	*F*, *F′*

[illegible] se draw the [illegible] of the equa[illegible] y and find [illegible] the curve is [illegible]



[illegible] values of x are [illegible] axis of x. If, [illegible] equal, and [illegible]

[illegible] two values of [illegible] 2, 1). Similarly [illegible] to any value [illegible] ordinate paper [illegible]

2. Plot the equation $x^2 + y^2 = 25$.

We may write this $y^2 = 25 - x^2$.

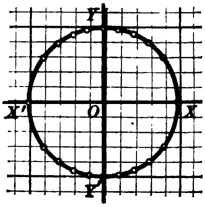
If $x^2 > 25$, or if $x > 5$, y is imaginary, and similarly if $x < -5$. We will first take for x values from -4 to $+4$.

If $x =$	-4	-3	-2	-1	0	1	2	3	4
Then $y =$	± 3	± 4	± 4.6	± 4.9	± 5	± 4.9	± 4.6	± 4	± 3

Furthermore, if $x = \pm 5$, $y = 0$, which gives two other points.

Plotting the points and drawing the smooth curve as before, the graph is seen to be a *circle* (circumference). The graph of a quadratic equation in the form $x^2 + y^2 = r^2$ is always a circle.

There is a more general form for the equation of a circle, $x^2 + y^2 + ax + by + c = 0$, where a, b, and c are any given numbers.



3. Plot the equation $4x^2 + 9y^2 = 288$.

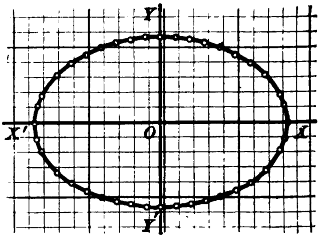
We may write this $y^2 = 32 - \frac{4}{9}x^2$.

If $\frac{4}{9}x^2 > 32$, or if $\frac{2}{3}x > \sqrt{32}$, or $\frac{2}{3}x < -\sqrt{32}$, y is imaginary; that is, x cannot be greater than $\frac{3}{2}\sqrt{32}$, or about 8.5, and x cannot be less than -8.5.

If $x =$	± 8.3	± 8	± 7	± 6	± 5	± 4	± 3	± 2	± 1
Then $y =$	± 1.2	± 1.9	± 3.2	± 4.0	± 4.6	± 5.0	± 5.3	± 5.5	± 5.6

Furthermore, if $x = \pm 8.5$, $y = 0$; and if $x = 0$, $y = \pm 5.7$.

It will be seen that $+8.3$ corresponds both to $+1.2$ and to -1.2, and that -8.3 also corresponds both to $+1.2$ and to -1.2.

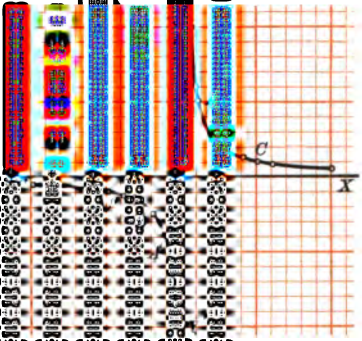


Plotting the points and drawing the curve, the graph is an *ellipse*. The graph of a quadratic equation in the form $ax^2 + by^2 = c$ is always an ellipse, a, b, and c having any values, and a and b having like signs. For example, $2x^2 + 3y^2 - 18 = 0$ is the equation of an ellipse.

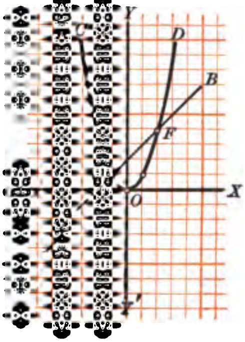
[illegible] bles:

[illegible]	6	10
[illegible]	$\frac{5}{6}$	$\frac{1}{2}$

[illegible]	-6	-10
[illegible]	$-\frac{5}{6}$	$-\frac{1}{2}$



[illegible] e see there- [illegible] uation may [illegible]



Exercise 169. Graphs

Examples 1 to 5, oral — Examples 6 to 29, written

1. What kind of graph has the equation $x + 2y = 7$?

2. What kind of curve is represented by $x^2 + y^2 = 16$?

3. What kind of conic is represented by $y^2 = 10x$?

4. What kind of conic is represented by $x^2 + 5y^2 = 16$?

5. What kind of conic is represented by $xy = 25$?

Plot the following equations:

6. $x^2 + y^2 = 9.$

7. $x^2 + 4y^2 = 9.$

8. $x^2 = 9y.$

9. $x^2 + 4x + 3 = y.$

10. $x^2 + 5x + 4 = y.$

11. $xy = 4.$

12. $x^2 - 5x + 6 = y.$

13. $x^2 + 2y^2 = 16.$

Find by graphs the roots of the following by letting the first member equal y, and then considering the graph when $y = 0$:

14. $x^2 - 7x + 12 = 0.$

15. $x^2 - 9x + 20 = 0.$

16. $x^2 - 13x + 42 = 0.$

17. $x^2 + 3x - 10 = 0.$

18. $x^2 + 4x - 21 = 0.$

19. $x^2 - 4x - 21 = 0.$

Solve exactly or approximately by the use of graphs:

20. $x^2 + y^2 = 169$
$3x - 2y = -9$

21. $x^2 + y^2 = 100$
$5x + y = 46$

22. $x^2 + y^2 = 100$
$3x + 4y = 50$

23. $x^2 + 9y^2 = 81$
$x - 3y = 5$

24. $2x + y = 5$
$3x^2 - 7y^2 = 5$

25. $xy = 6$
$x - y = 0$

26. $x^2 = 4y$
$x + 2y = 6$

27. $xy = 24$
$x + 2y = 14$

28. $x^2 = 9y$
$2x + y = 16$

29. $x^2 + x + 3 = y$
$x - 2y + 4 = 0$

[illegible] Given two [illegible] are all of [illegible]

[illegible] (1)

[illegible] (2)

[illegible] 3y.

[illegible] for x in (2),

[illegible] 69,

[illegible]

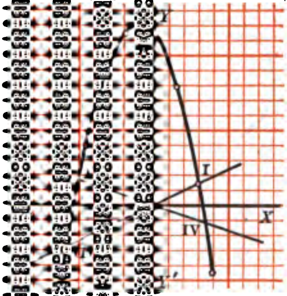
$[illegible]\ y = \frac{3 \pm \sqrt{285}}{6}.$

$[illegible] = \frac{-3 \mp \sqrt{285}}{2}.$

$[illegible] \frac{3 + \sqrt{285}}{2},$

$[illegible] \frac{\sqrt{285}}{6}.$

[illegible]



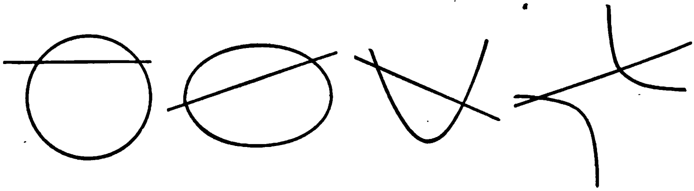
[illegible]	[illegible]	[illegible]	[illegible]	8	9
[illegible]	[illegible]	[illegible]	[illegible]	− 9	− 16

[illegible] each square [illegible] pairs of roots [illegible], in order.

Exercise 170. Simultaneous Quadratics

Examples 1 to 10, oral — Examples 11 to 18, written

1. In how many points can a straight line intersect a circle? intersect an ellipse? a parabola? an hyperbola?



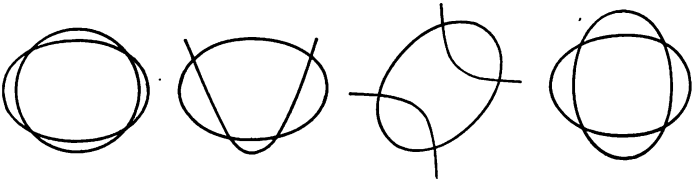
2. How many roots in the system $x + 2y = 7$, $x^2 + y^2 = 16$?

3. How many roots in the system $x - 4y = 5$, $x^2 + 3y^2 = 9$?

4. How many roots in the system $x = 2y$, $y^2 = 4x$?

5. How many roots in the system $3x + y = 9$, $xy = 6$?

6. In how many points can an ellipse intersect a circle? intersect a parabola? an hyperbola? another ellipse?



7. How many roots in the system $x^2 + 5y^2 = 16$, $x^2 + y^2 = 9$?

8. How many roots in the system $x^2 + 5y^2 = 16$, $y^2 = x + 3$?

9. How many roots in the system $x^2 + 5y^2 = 16$, $xy = 1$?

10. How many roots in the system $5x^2 + y^2 = 16$, $x^2 + 5y^2 = 16$?

Solve and plot the equations in the following examples:

11. Ex. 2.

12. Ex. 3.

13. Ex. 4.

14. Ex. 5.

15. Ex. 7.

16. Ex. 8.

17. Ex. 9.

18. Ex. 10.

282. Two Equations Homogeneous except for Absolute Terms. If two quadratic equations are homogeneous except for the absolute terms, they can also be solved. For if we eliminate the absolute term from either equation, we have a case exactly like the one discussed on page 343.

Solve the equations

$$x^2 + xy + 2y^2 = 44 \qquad (1)$$

$$2x^2 - xy + y^2 = 16 \qquad (2)$$

Multiplying (1) by 4, $4x^2 + 4xy + 8y^2 = 176.$ (3)

Multiplying (2) by 11, $22x^2 - 11xy + 11y^2 = 176.$ (4)

Dividing (4) – (3) by 3, $6x^2 - 5xy + y^2 = 0.$

Factoring, $(3x - y)(2x - y) = 0.$

From $3x - y = 0$ we have $y = 3x.$

From $2x - y = 0$ we have $y = 2x.$

Substituting $3x$ for y in (2),

$$2x^2 - 3x^2 + 9x^2 = 16.$$

$$8x^2 = 16.$$

$$x^2 = 2.$$

$$x = \pm\sqrt{2}.$$

$$y = 3x = \pm 3\sqrt{2}.$$

Substituting $2x$ for y in (2),

$$2x^2 - 2x^2 + 4x^2 = 16.$$

$$4x^2 = 16.$$

$$x^2 = 4.$$

$$x = \pm 2.$$

$$y = 2x = \pm 4.$$

Therefore, when $x = +\sqrt{2},\ -\sqrt{2},\ +2,\ -2,$
the corresponding value of $y = +3\sqrt{2},\ -3\sqrt{2},\ +4,\ -4.$

Exercise 171. Simultaneous Quadratics

All work written

Solve the following equations:

1. $x^2 - xy + y^2 = 21$
   $x^2 + 3xy - 2y^2 = 38$

2. $x^2 - xy + y^2 = 21$
   $y^2 - 2xy = -15$

3. $x^2 + 2xy - y^2 = 50$
   $3x^2 - xy + y^2 = 75$

4. $x^2 + 7xy + y^2 = 45$
   $5x^2 - xy + 2y^2 = 33$

5. $2x^2 - xy + 2y^2 = 12$
   $3x^2 - 2xy + 4y^2 = 20$

6. $3x^2 + xy + 5y^2 = 81$
   $2x^2 - 3xy + 4y^2 = 27$

283. Special Forms of Quadratics. The two cases mentioned on pages 343–345 are the only ones likely to be met that can always be solved by quadratic methods. Special forms will arise that can be so solved, sometimes involving equations of degree higher than the second, and sometimes involving linear equations. Such cases usually require only a little ingenuity in treatment.

1. Solve the equations

$$x^3 + y^3 = 9 \tag{1}$$

$$x + y = 3 \tag{2}$$

Dividing (1) by (2), $x^2 - xy + y^2 = 3.$ (3)

From (2), $y = 3 - x$

Substituting $3 - x$ for y in (3), $3x^2 - 9x + 6 = 0,$

or $x^2 - 3x + 2 = 0,$

or $(x - 2)(x - 1) = 0.$

$$\therefore\ x = 2 \text{ or } 1,$$

$$y = 1 \text{ or } 2.$$

Check. $2^3 + 1^3 = 9$, and $2 + 1 = 3$;

$1^3 + 2^3 = 9$, and $1 + 2 = 3$.

2. Solve the equations

$$x^2 + y^2 + x + y = 18 \tag{1}$$

$$xy = 6 \tag{2}$$

From (2), $2xy = 12.$

Adding this to (1), $x^2 + 2xy + y^2 + x + y = 30,$

or $(x + y)^2 + (x + y) - 30 = 0.$

Factoring $[(x + y) + 6][(x + y) - 5] = 0.$

Therefore $x + y = -6$, and $y = -6 - x$,

or $x + y = 5$, and $y = 5 - x$.

Substituting these values of y in (2),

$-6x - x^2 = 6.$	$5x - x^2 = 6.$
$x^2 + 6x + 6 = 0.$	$x^2 - 5x + 6 = 0.$
$\therefore\ x = -3 \pm \sqrt{3}.$	$\therefore\ x = 2 \text{ or } 3.$
$\therefore\ y = -3 \mp \sqrt{3}.$	$\therefore\ y = 3 \text{ or } 2.$

We may check by substituting in equation (1).

3. Solve the equations

$$\frac{1}{x}+\frac{1}{y}=\frac{3}{2} \qquad (1)$$

$$\frac{1}{x^2}+\frac{1}{y^2}=\frac{5}{4} \qquad (2)$$

From (1), $\frac{1}{y}=\frac{3}{2}-\frac{1}{x}$.

Substituting for $\frac{1}{y}$ in (2), $\frac{1}{x^2}+\frac{9}{4}-\frac{3}{x}+\frac{1}{x^2}=\frac{5}{4}$,

whence $\frac{2}{x^2}-\frac{3}{x}+1=0$.

Solving, $x = 2$ or 1.

Substituting 2 for x, $\frac{1}{y}=\frac{3}{2}-\frac{1}{x}=\frac{3}{2}-\frac{1}{2}=1$,

whence $y = 1$.

Substituting 1 for x, $\frac{1}{y}=\frac{3}{2}-1=\frac{1}{2}$,

whence $y = 2$.

Therefore $x = 2$ when $y = 1$, and $x = 1$ when $y = 2$.

284. Symmetric Equations. When two unknown quantities can be interchanged without altering an equation, the equation is said to be *symmetric* with respect to the two quantities.

For example, in the equation $x + y = 5$ we may interchange x and y without altering the equation, since $y + x = 5$ is evidently the same as $x + y = 5$. But the equation $x + 2y = 5$ is not symmetric, for it is not the same as $y + 2x = 5$. The equations in Exs. 1–3 above are all symmetric. Two symmetric *quadratic* equations can always be solved by quadratic methods.

285. Roots of Symmetric Equations. *In an equation symmetric with respect to x and y the values of x and y are always the same, although differently arranged.*

Since y may be put in place of x, in such cases we must get the same values for y as for x. It will be seen that this is so in Exs. 1–3 above. Therefore, having obtained the values of x, we may write down the values of y without solving, simply arranging them in proper pairs so they correspond and may be checked.

Exercise 172. Simultaneous Quadratics

Examples 1 to 4, oral — Examples 5 to 44, written

1. In the equations $x + y = 9$, and $x^2 + y^2 = 41$, if we find that $x = 5$ or 4, what do we know about the values of y? Why? How are they arranged by pairs?

2. In the equations $xy = 15$, $x + y = 8$, if we find that $x = 3$ or 5, what is the rest of the solution?

3. In the equations $\frac{1}{x} + \frac{1}{y} = \frac{5}{12}$, $\frac{1}{xy} = \frac{1}{24}$, if we find that $x = 4$ when $y = 6$, what are the other two values of x and y? Why?

4. How would you proceed to solve the equations $x^3 - y^3 = 98$, $x - y = 2$? Why is this better than substituting the value of y (in the form $y = x - 2$) in the first equation?

Solve and plot the following equations:

5. $x^2 - y = 3$
$x - y = -3$

6. $3y = 2x - 1$
$9y = x^2 + 2$

7. $3x - y = 5$
$3x + y^2 = 17$

8. $5x - 3y + 6 = 0$
$5x - 2y^2 + 5 = 0$

9. $3x + y - 12 = 0$
$4x^2 - 5x - y^2 = 12$

10. $4x - 13y + 11 = 0$
$x^2 - 2xy - 2y^2 = 22$

11. $2x - y - 7 = 0$
$x^2 - 7x - 2y^2 = -17$

12. $3x + y - 16 = 0$
$x^2 + 4y^2 = 29$

13. $x + y = 12$
$xy = 35$

14. $x - y = 22$
$xy = 663$

15. $x + y = 38$
$xy = 357$

16. $x^2 + y^2 = 661$
$x^2 - y^2 = 589$

17. $3x^2 - y^2 = 59$
$2x^2 + 3y^2 = 98$

18. $5x^2 + 2y^2 = 220$
$2x + 5y = 54$

19. $16x^2 - 9y^2 = 0$
$2xy - 5x + 6y = 33$

20. $5x^2 + y^2 = 126$
$5y^2 + x^2 = 30$

Solve the following equations:

21. $\dfrac{x+y}{x-y}=12$
$x^2-y^2=48$

22. $16xy-x^2y^2=60$
$x+y=7$

23. $x^2-y^2=228$
$xy-y^2=42$

24. $x=\frac{7}{3}\sqrt{x+y}$
$y=\frac{11}{3}\sqrt{x+y}$

25. $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{9}{20}$
$\dfrac{1}{x^2}+\dfrac{1}{y^2}=\dfrac{41}{400}$

26. $x^2+y^2-91=xy$
$x^2+y^2+xy=223$

27. $42(x^2+y^2)=85xy$
$15xy=2520$

28. $x^2+y^2+xy=67$
$x^2+y^2=53$

29. $x^3+y^3=65$
$x+y=5$

30. $x^3+y^3=1304$
$x+y=8$

31. $x^3+y^3=280$
$x^2-xy+y^2=28$

32. $x^2+xy=77$
$xy+y^2=44$

33. $\dfrac{1}{x}-\dfrac{1}{y}=\dfrac{1}{36}$
$xy^2-x^2y=324$

34. $x^2+xy=260$
$xy+y^2=140$

35. $xy+x=20$
$xy-y=12$

36. $x=6\sqrt{x+y}$
$y=2\sqrt{x+y}$

37. $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{2}$
$\dfrac{1}{x^2}+\dfrac{1}{y^2}=\dfrac{5}{36}$

38. $x^2-xy=45$
$xy-y^2=-36$

39. $x^2+xy=55$
$y^2+xy=66$

40. $x+xy+y=5$
$x^2+xy+y^2=7$

41. $x^3+y^3=37$
$x^2-xy+y^2=37$

42. $x^2+y=5(x-y)$
$x+y^2=2(x-y)$

43. $x^2-xy+y^2=103$
$x-xy+y=-79$

44. $x+\sqrt{xy}+y=14$
$x^2+xy+y^2=84$

Exercise 173. Problems involving Simultaneous Quadratics

All problems for written work. Reject meaningless results

1. The sum of two numbers is 32 and their product is 255. Find the numbers.

2. The difference of two numbers is 4 and their product is 437. Find the numbers.

3. The sum of two numbers is 24 and the sum of their reciprocals is $\frac{3}{16}$. Find the numbers.

4. The sum of two numbers is 74 and the sum of their square roots is 12. Find the numbers.

5. The sum of two numbers is 23 and the sum of their cubes is 3059. Find the numbers.

6. The difference of the cubes of two numbers is 218 and the difference of the numbers is 2. Find the numbers.

7. The sum of the squares of two numbers is 89 and the product of the numbers is 40. Find the numbers.

8. The sum of two numbers added to the sum of their squares is 686, and the difference of the numbers added to the difference of their squares is 74. Find the numbers.

9. The product of two numbers is 91 greater than ten times the first number, and 51 greater than ten times the second number. Find the numbers.

10. There are two numbers formed by the same two digits in reverse order. The sum of the numbers is 55 times the difference between the two digits, and the difference between the squares of the two numbers is 1980. Find the numbers.

Let $10x + y$ and $10y + x$ represent the numbers.

11. A strip of cloth when wet shrinks $12\frac{1}{2}\%$ in length and $6\frac{1}{4}\%$ in width. If the number of square yards is diminished $11\frac{1}{2}$ when it is wet, and if the sum of the length and width is $8\frac{1}{16}$ yd. less than before, what were the dimensions of the strip before it was wet?

12. A workman wishes to enlarge a drawing of a rectangle so that the area will be twice the original area, keeping the ratio of the length to the width unchanged. The original drawing is 8 in. by 10 in. Find the sides of the enlarged rectangle.

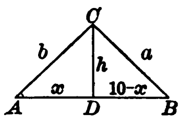
13. A rectangular field is 30 yd. wide, and the length exceeds the width by $66\frac{2}{3}\%$. How much must the width be decreased and the length increased so that the area will remain the same while the perimeter is increased 30 yd.?

14. Two points move, each at a uniform rate, on the sides of the right angle of a right triangle *ABC*, away from the vertex *A*, starting from the two points *P* and *Q*, 3 in. and 4 in. respectively from the vertex *A*. After 2 sec. they are 10 in. apart, and after 6 sec. they are 20 in. apart. Find the rate of each.

15. There are two lines such that if they are made the sides of the right angle of a right triangle the hypotenuse will be 35 in.; but if one is made the hypotenuse and the other is made a side, the square on the other side will be 343. Find their lengths.

16. A boat's crew, rowing at half their usual rate, row 2 mi. down a river and back in 1 hr. 40 min. At their usual rate in still water they would have gone over the same course in 40 min. Find their rate of rowing in still water and the rate of the current.

17. In the figure below, $c = AB = 10, b = 8, a = 7$. Required to find the length of the perpendicular *CD*.

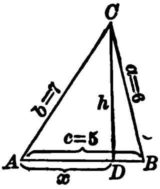


We have $x^2 + h^2 = b^2$, and $\overline{DB}^2 + h^2 = a^2$. But $\overline{DB} = 10 - x$. Therefore $x^2 + h^2 = 64$, and $(10-x)^2 + h^2 = 49$. Hence we have two quadratics.

18. In Ex. 17 suppose $c = 15$, $b = 10$, and $a = 9$. Find the length of *CD*, and then find the area of the triangle.

19. Two cubes have together the volume 407 cu. in., and the sum of one edge of the one and one edge of the other is 11 in. Find the volume of each.

20. In the figure below, $c = AB = 5$, $b = 7$, $a = 6$. Find the length of the perpendicular CD.



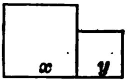
21. In Ex. 20 suppose $c = 7$, $b = 9$, $a = 8$. Find the length of CD and the area of the triangle.

22. As in Exs. 18 and 20, find the area of the triangle whose sides are 11, 12, and 13.

23. Draw a rectangle whose perimeter is 20 in. and whose area is $22\frac{3}{4}$ sq. in.

24. Draw a rectangle whose area is $23\frac{5}{8}$ sq. in. and whose base is $3\frac{1}{4}$ in. greater than the altitude.

25. Two squares are placed together as here shown, forming a figure of six sides. The area of the figure is 52 sq. in. and its perimeter is 32 in. Find the length of side in each square.



26. The altitude of a trapezoid is 18 ft. Its area is equal to that of a rectangle with sides equal to the parallel bases of the trapezoid. Three times the smaller base added to the larger base is four times the altitude of the trapezoid. Find the two bases.

27. The sum of two numbers is 11, and the cube of their sum exceeds the sum of their cubes by 792. Find the numbers.

28. A number is formed by two digits. The second digit is less by 8 than the square of the first digit. If 9 times the first digit is added to the number, the order of digits is reversed. Find the number.

29. A number is formed by three digits, the third digit being the sum of the other two. The product of the first and third digits exceeds the square of the second by 5. If 396 is added to the number, the order of the digits is reversed. Find the number.

30. If the product of two numbers is increased by their sum, the result is 79. If their product is diminished by their sum, the result is 47. Find the numbers.

CHAPTER XIX

POWERS AND ROOTS COMPLETED

286. Review of the Meaning of a^0 and a^{-n}. As already explained (§ 252), $a^0 = 1$, and (§ 251) $a^{-n} = \frac{1}{a^n}\cdot$ The reasonableness of this is further seen from the following relations:

Since	a^3	means	aaa,	or	$a^4 \div a$,
and	a^2	means	aa,	or	$a^3 \div a$,
it should follow that	a^1	means	a,	or	$a^2 \div a$,
and that	a^0	means	1,	or	$a \div a$,
and that	a^{-1}	means	$\frac{1}{a}$,	or	$1 \div a$,
and so on, so that	a^{-n}	means	$\frac{1}{a^n}$.		

287. Review of the Meaning of $a^{\frac{1}{n}}$. As already explained (§ 239), $a^{\frac{1}{n}} = \sqrt[n]{a}$. The reasonableness of this is further seen from the following relations:

Starting with a^4,

if this exponent is half as large, we have a^2, or $\sqrt{a^4}$;

if this exponent is half as large, we have a, or $\sqrt{a^2}$;

if this exponent is half as large, $a^{\frac{1}{2}}$ should equal $\sqrt{a}$;

and in the same way $a^{\frac{1}{n}}$ should equal $\sqrt[n]{a}$, and

$$a^{\frac{m}{n}} \text{ should equal } \sqrt[n]{a^m}.$$

We thus see that not only do a^0, a^{-n}, $a^{\frac{m}{n}}$, have the meanings assigned to them because they obey the fundamental laws of exponents (§§ 204, 239, 240), but because these meanings fit into the general definitions.

288. Laws of Exponents. The following laws have already been proved for positive integral exponents (§ 204) and assumed for other kinds of exponents (§§ 239 and 251):

Law I. $a^m a^n = a^{m+n}$.

Law II. $a^m \div a^n = a^{m-n}$.

Law III. $(ab)^n = a^n b^n$.

Law IV. $(a^m)^n = a^{mn}$.

In any of these cases m or n, or both m and n, may be either fractional or negative, or both fractional and negative, as well as positive and integral. To prove all possible cases would take too much time, but a few typical ones will now be considered. All of the others admit of similar treatment.

It is unnecessary to master all of the proofs here given, and the teacher may safely omit one or more. The important thing is that the student should know that the fundamental laws hold for all kinds of exponents.

1. To prove that $a^{\frac{m}{n}} = a^{\frac{pm}{pn}}$.

To prove this we must first recall that $\frac{m}{n}$ is here a symbol meaning that the nth root of the mth power is to be taken, and that it is not a fraction in the ordinary sense. We are now to show that it may be reduced like an ordinary fraction, and we are to show this by the laws of positive integral exponents already proved.

Let $x = a^{\frac{m}{n}}$.

Then $x^n = a^m$, Axiom 5

and $x^{pn} = a^{pm}$. Axiom 5

Therefore, extracting roots, $x = a^{\frac{pm}{pn}}$.

$\therefore a^{\frac{m}{n}} = a^{\frac{pm}{pn}}$, since each equals x.

2. To prove that $(ab)^m = a^m b^m$ when m is negative ($m = -n$), that is, that $(ab)^{-n} = a^{-n}b^{-n}$.

We have, by the definition of negative exponent and by a law already proved,

$$(ab)^{-n} = \frac{1}{(ab)^n} = \frac{1}{a^n b^n} = a^{-n}b^{-n}.$$

3. To prove that $(ab)^{\frac{p}{q}} = a^{\frac{p}{q}}b^{\frac{p}{q}}$.

Let $x = (ab)^{\frac{p}{q}}$.

Then $x^q = (ab)^p = a^pb^p$. Why?

$\therefore\ x = \sqrt[q]{a^p} \cdot \sqrt[q]{b^p} = a^{\frac{p}{q}}b^{\frac{p}{q}}$. Why?

$\therefore\ (ab)^{\frac{p}{q}} = a^{\frac{p}{q}}b^{\frac{p}{q}}$. Why?

4. To prove that $a^{-p}a^{-q} = a^{-p-q}$.

$$a^{-p}a^{-q} = \frac{1}{a^p} \cdot \frac{1}{a^q} = \frac{1}{a^{p+q}} = a^{-p-q}.$$ Explain

5. To prove that $a^{\frac{m}{n}}a^{\frac{p}{q}} = a^{\frac{mq+np}{nq}}$.

We have $a^{\frac{m}{n}}a^{\frac{p}{q}} = a^{\frac{mq}{nq}}a^{\frac{np}{nq}}$, as in 1

$= (a^{mq}a^{np})^{\frac{1}{nq}}$, as in 3

$= (a^{mq+np})^{\frac{1}{nq}}$ Why?

$= a^{\frac{mq+np}{nq}}$. Why?

6. To prove that $(a^{-m})^{-n} = a^{mn}$.

We have $(a^{-m})^{-n} = \left(\frac{1}{a^m}\right)^{-n} = (a^m)^n = a^{mn}$. Explain.

7. To prove that $(a^{\frac{m}{n}})^{\frac{p}{q}} = a^{\frac{mp}{nq}}$.

Let $x = (a^{\frac{m}{n}})^{\frac{p}{q}}$.

Then $x^q = (a^{\frac{m}{n}})^p$ Why?

$= a^{\frac{m}{n}}a^{\frac{m}{n}} \cdots p$ factors,

$= a^{\frac{m+m+\cdots p \text{ terms}}{n}}$, as in 5.

$\therefore\ x^{nq} = a^{mp}$, Why?

and $x = a^{\frac{mp}{nq}}$.

$\therefore\ (a^{\frac{m}{n}})^{\frac{p}{q}} = a^{\frac{mp}{nq}}$, since each equals x.

289. Advantage of Negative and Fractional Exponents. The advantage of using a^{-n} instead of $\frac{1}{a^n}$, and $a^{\frac{3}{4}}$ instead of $\sqrt[4]{a^3}$, already mentioned in §§ 242 and 251, should be fully appreciated.

Thus it is easier to see that $(a^{-\frac{3}{4}})^{-\frac{4}{3}} = a$, than that

$$\frac{1}{\sqrt[3]{\left(\frac{1}{\sqrt[4]{a^3}}\right)^4}} = a,$$

although the two mean the same thing.

Exercise 174. Negative and Fractional Exponents

Examples 1 to 5, oral — Examples 6 to 44, written

1. Simplify $(a^{-2})^{-3}$; $(a^{\frac{1}{2}})^2$; $(a^{\frac{2}{3}})^3$; $(m^{\frac{2}{3}})^{\frac{3}{2}}$.
2. Expand $(a^{-1} + b^{-1})^2$; $(a^{-2} - b^{-2})^2$; $(a^{-2} + a^2)^2$.
3. Multiply $a^{-1} + b^{-1}$ by $a^{-1} - b^{-1}$; by $2a^{-1}$.
4. Divide $a^{-2} - b^{-2}$ by $a^{-1} + b^{-1}$; by $a^{-1} - b^{-1}$.
5. Expand $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2$; $(a^{\frac{1}{2}} - b^{\frac{1}{2}})^2$; $(a^{\frac{1}{2}} + b^{-1})^2$.

Perform the operations indicated:

6. $(x^{-2} + x^{-1} + 1)(x^{-2} - x^{-1} + 1)$.
7. $(a^{\frac{2}{3}} + b^{\frac{2}{3}})(a^{\frac{2}{3}} - b^{\frac{2}{3}})$.
8. $(x^2 + x^{-2})(x^2 - x^{-2})$.
9. $(a^{-\frac{1}{2}} + b^{-\frac{1}{2}})(a^{-\frac{1}{2}} - b^{-\frac{1}{2}})$.
10. $(p^2q^2 + q^{-2})(p^2q^2 - q^{-2})$.
11. $(a^{\frac{p}{q}} + a^{\frac{m}{n}})^2$.
12. $(a^{-\frac{p}{q}} + a^{\frac{p}{q}})^3$.
13. $(a^{-\frac{p}{q}} + a^{\frac{p}{q}})(a^{-\frac{p}{q}} - a^{\frac{r}{q}})$.
14. Divide $x^{-10} - x^{-5} + 1$ by $x^{-2} - x^{-1} + 1$.
15. Divide $a^{-3} + 2a^{-2}b^{-1} - 3b^{-3}$ by $a^{-1} - b^{-1}$.
16. Express $\sqrt[3]{x^2} + 2\sqrt[3]{xy} + \sqrt[3]{y^2}$ with fractional exponents and then extract the square root.
17. Express $\sqrt[3]{\frac{1}{a^4}} + 2 + \sqrt[3]{a^4}$ with fractional exponents and then extract the square root.

Perform the following multiplications:

18. $x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$ by $x^{\frac{1}{3}} + y^{\frac{1}{3}}$.

19. $x^{\frac{3}{4}} + x^{\frac{1}{2}} + x^{\frac{1}{4}} + 1$ by $x^{\frac{1}{2}} + x^{\frac{1}{4}} + 1$.

20. $8x^{\frac{3}{2}} - 4xy^{\frac{1}{2}} + 2x^{\frac{1}{2}}y - y^{\frac{3}{2}}$ by $2x^{\frac{1}{2}} - 3y^{\frac{1}{2}}$.

21. $x - 6a^{\frac{1}{3}}x^{\frac{2}{3}} + 12a^{\frac{2}{3}}x^{\frac{1}{3}} - 8a$ by $x^{\frac{2}{3}} - 4a^{\frac{1}{3}}x^{\frac{1}{3}} + 4a^{\frac{2}{3}}$.

22. $x^{-3} + x^{-2} + x^{-1} + 1$ by $x^{-2} + x^{-1} + 1$.

23. $x^{-\frac{3}{2}} + 3x^{-1} + 3x^{-\frac{1}{2}} + 1$ by $x^{-1} + 2x^{-\frac{1}{2}} + 1$.

24. $x^{-\frac{3}{2}} - 6a^{\frac{1}{2}}x^{-1} + 12ax^{-\frac{1}{2}} - 8a^{\frac{3}{2}}$ by $x^{-1} - 4a^{\frac{1}{2}}x^{-\frac{1}{2}} - 4a$.

Perform the following divisions:

25. $acx^2y^{-1} - bcyx^{-1}$ by $cx^{-\frac{1}{2}}y^{-\frac{1}{2}}$.

26. $a^2 - 3a^{\frac{5}{3}} + 6a^{\frac{4}{3}} - 7a + 6a^{\frac{2}{3}} - 3a^{\frac{1}{3}} + 1$ by $a^{\frac{2}{3}} - a^{\frac{1}{3}} + 1$.

27. $\frac{9}{16}a^2 - \frac{7}{8}a^{\frac{3}{2}}b^{\frac{1}{2}} + \frac{19}{36}ab + \frac{1}{6}a^{\frac{1}{2}}b^{\frac{3}{2}}$ by $\frac{3}{2}a^{\frac{1}{2}} + \frac{1}{3}b^{\frac{1}{2}}$.

28. $x^{-\frac{7}{3}} - 3x^{-2} + x^{-\frac{5}{3}} - 4 + 12x^{-\frac{1}{3}} - 4x^{-\frac{2}{3}}$ by $x^{-\frac{5}{3}} - 4$.

Find the square roots of the following expressions:

29. $a^{\frac{4}{3}}b^{-1} + 4ab^{-\frac{1}{2}} - 2a^{\frac{2}{3}} - 12a^{\frac{1}{3}}b^{\frac{1}{2}} + 9b$.

30. $60mn^{\frac{5}{4}} - 4m^{\frac{4}{3}}n^{\frac{3}{2}} - 48m^{\frac{5}{3}}n^{\frac{7}{4}} + 16m^2n^2 + 25m^{\frac{2}{3}}n$.

31. $9x^{-\frac{4}{3}} + 24x^{-1}y^{-\frac{1}{3}} + 46x^{-\frac{2}{3}}y^{-\frac{2}{3}} + 40x^{-\frac{1}{3}}y^{-1} + 25y^{-\frac{4}{3}}$.

32. $a^{-4} + b^{-6} + c^{-8} + 2a^{-2}b^{-3} + 2a^{-2}c^{-4} + 2b^{-3}c^{-4}$.

Rationalize the denominator of the following fractions:

33. $\dfrac{3}{5^{\frac{1}{2}} - 2^{\frac{1}{2}}}$.

34. $\dfrac{3}{10^{\frac{1}{2}} - 6^{\frac{1}{2}}}$.

35. $\dfrac{6}{12^{\frac{1}{2}} + 5^{\frac{1}{2}}}$.

36. $\dfrac{a^{\frac{1}{2}} + b^{\frac{1}{2}}}{a^{\frac{1}{2}} - b^{\frac{1}{2}}}$.

37. $\dfrac{\sqrt{11} + \sqrt{5}}{\sqrt{11} - \sqrt{5}}$.

38. $\dfrac{1 + 4\sqrt{5}}{6 + 2\sqrt{3}}$.

39. $\dfrac{\sqrt{a} - \sqrt{3b}}{\sqrt{a} + \sqrt{3b}}$.

40. $\dfrac{\sqrt{2x} + \sqrt{7}}{\sqrt{2x} - \sqrt{7}}$.

41. $\dfrac{30}{2 - \sqrt{3} + \sqrt{5}}$.

42. $\dfrac{12}{3 + 2^{\frac{1}{2}} - 3^{\frac{1}{2}}}$.

43. $\dfrac{a}{b + (a + b)^{\frac{1}{2}}}$.

44. $\dfrac{x + \sqrt{x - y}}{x - \sqrt{x - y}}$.

290. The Square Root of a Binomial Surd. Since $(\sqrt{a}+\sqrt{b})^2 = a + 2\sqrt{ab} + b$, if we can write the binomial surd $x + \sqrt{y}$ in the form $a + 2\sqrt{ab} + b$ we can find the square root by inspection.

For example, find the square root of $7 + 4\sqrt{3}$.

We may write $$7 + 4\sqrt{3} = 7 + 2\sqrt{12}$$
$$= 4 + 2\sqrt{4 \cdot 3} + 3 = (2 + \sqrt{3})^2.$$

Therefore $$\sqrt{7 + 4\sqrt{3}} = 2 + \sqrt{3}.$$

To find the square root of a binomial surd, write it in the form $a + 2\sqrt{ab} + b$*. Then the square root will be in the form* $\sqrt{a} + \sqrt{b}$.

That is, $$\sqrt{a + 2\sqrt{ab} + b} = \sqrt{a} + \sqrt{b}.$$

Exercise 175. Square Root of a Binomial Surd

Examples 1 to 3, oral — Examples 4 to 27, written

1. Square $2 + \sqrt{3}$; $\sqrt{2} + \sqrt{3}$; $\sqrt{2} + 3$; $\sqrt{2} - \sqrt{3}$.
2. Square $2\sqrt{3}$; $2 + 2\sqrt{3}$; $3 + 2\sqrt{3}$; $3 - 2\sqrt{3}$.
3. Square $a + \sqrt{b}$; $a + 2\sqrt{b}$; $\sqrt{a} + \sqrt{b}$; $\sqrt{a} - 2\sqrt{b}$.

Find the square roots of the following expressions:

4. $5 + 2\sqrt{6}$.
5. $11 + 6\sqrt{2}$.
6. $21 - 12\sqrt{3}$.
7. $17 + 12\sqrt{2}$.
8. $9 - 4\sqrt{5}$.
9. $6 - 2\sqrt{5}$.
10. $8 + 2\sqrt{15}$.
11. $36 + 18\sqrt{3}$.
12. $30 + 12\sqrt{6}$.
13. $23 - 4\sqrt{15}$.
14. $9 + 2\sqrt{14}$.
15. $12 + 2\sqrt{35}$.

Solve the following equations:

16. $x^2 = 12 + 2\sqrt{11}$.
17. $x^2 = 18 + 8\sqrt{2}$.
18. $x^2 = 27 + 10\sqrt{2}$.
19. $x^2 = 51 - 14\sqrt{2}$.
20. $x^2 = 66 - 16\sqrt{2}$.
21. $2(x^2 - 2) = \sqrt{15}$.
22. $x^2 - 16 = 2\sqrt{55}$.
23. $x^2 - 11 = 4\sqrt{7}$.
24. $x^2 - 23 = 8\sqrt{7}$.
25. $x^2 - 32 = 10\sqrt{7}$.
26. $x^2 - 15 = 4\sqrt{11}$.
27. $x^2 + 14\sqrt{11} = 60$.

291. Cube Root of a Polynomial. If we cube $a + b$, the result is, as shown in § 95, $a^3 + 3a^2b + 3ab^2 + b^3$. In extracting the cube root we reverse this process of cubing, thus:

$$\begin{array}{ll}
 & a^3 + 3a^2b + 3ab^2 + b^3 \;\underline{\big|\, a + b} = \text{Root} \\
 & \underline{a^3} \\
3a^2 & \big|\; 3a^2b + 3ab^2 + b^3 = \text{First remainder} \\
\underline{3a^2 + 3ab + b^2} & \big|\; \underline{3a^2b + 3ab^2 + b^3} = b(3a^2 + 3ab + b^2)
\end{array}$$

The first term of the root is evidently a, because the cube of a is a^3.

Since in cubing a binomial we have the cube of the first term plus three times the product of the square of the first by the second, etc., we have in $3a^2b$ three times the product of the square of a by the second term. We therefore divide by $3a^2$ to find the second term, b.

In cubing $a + b$ we have $a^3 + b(3a^2 + 3ab + b^2)$. We therefore add $3ab + b^2$ to $3a^2$ and multiply $3a^2 + 3ab + b^2$ by b, thus completing the cube of $a + b$.

1. Find the cube root of $27x^3 - 189x^2y + 441xy^2 - 343y^3$.

$$\begin{array}{ll}
 & 27x^3 - 189x^2y + 441xy^2 - 343y^3 \;\underline{\big|\, 3x - 7y} \\
 & \underline{27x^3} \\
27x^2 & \big|\; -189x^2y + 441xy^2 - 343y^3 \\
\underline{27x^2 - 63xy + 49y^2} & \big|\; \underline{-189x^2y + 441xy^2 - 343y^3}
\end{array}$$

Here $3(3x)^2 = 27x^2$, corresponding to $3a^2$ in $(a + b)^3$.

To $27x^2$ is added $3(3x)(-7y) + (-7y)^2$, or $-63xy + 49y^2$.

This work should be compared, step by step, with the example above explained, the following questions being answered:

How is the first term of the root found?

Why is $27x^2$ taken as the divisor?

How is the second term found?

Why is $-63xy + 49y^2$ added to $27x^2$?

Why is $27x^2 - 63xy + 49y^2$ multiplied by $-7y$?

2. Find the cube root of $300x^2y - 5x(25x^2 + 48y^2) + 64y^3$. Rearranging, the work appears as follows:

$$\begin{array}{ll}
 & -125x^3 + 300x^2y - 240xy^2 + 64y^3 \;\underline{\big|\, -5x + 4y} \\
 & \underline{-125x^3} \\
75x^2 & \big|\; 300x^2y - 240xy^2 + 64y^3 \\
\underline{75x^2 - 60xy + 16y^2} & \big|\; \underline{300x^2y - 240xy^2 + 64y^3}
\end{array}$$

In case there are more than two terms in the root the method is essentially the same. After two terms have been found, their sum is considered to be a in the typical form $a^3 + 3a^2b + 3ab^2 + b^3$.

Thus if the part of the root already found is $x^2 - x$, we subtract $(x^2 - x)^3$, and the remainder is in the form $3a^2b + 3ab^2 + b^3$, in which $x^2 - x$ is a. We therefore divide by $3(x^2 - x)^2$ to find b, the next term. We proceed in a similar manner if there are more than three terms in the root, always letting a represent the part of the root already found.

3. Find the cube root of $x^6 - 3x^5 + 5x^3 - 3x - 1$.

$$\begin{array}{l|l|l}
 & x^6 - 3x^5 + 5x^3 - 3x - 1 & x^2 - x - 1 \\
 & x^6 & \\
\hline
3x^4 & -3x^5 \qquad\quad + 5x^3 & \\
3x^4 - 3x^3 + x^2 & -3x^5 + 3x^4 - x^3 & \\
\hline
3x^4 - 6x^3 + 3x^2 & -3x^4 + 6x^3 - 3x - 1 & \\
3x^4 - 6x^3 + 3x + 1 & -3x^4 + 6x^3 - 3x - 1 & \\
\hline
\end{array}$$

The first term of the root is x^2.

The first divisor is $3(x^2)^2$, or $3x^4$; and $-3x^5 \div 3x^4 = -x$, the second term.

Then $3(x^2)^2 + 3(x^2)(-x) + (-x)^2 = 3x^4 - 3x^3 + x^2$. This, multiplied by $-x$, equals $-3x^5 + 3x^4 - x^3$, thus completing the cube of $x^2 - x$.

Subtracting, and considering $x^2 - x$ as the first part of the root, a, we have subtracted the cube of the first part of the root, a^3. The remainder therefore contains $3a^2b + 3ab^2 + b^3$, where b is the next term. We therefore divide this remainder by $3a^2$, or $3(x^2 - x)^2$. Practically we need divide only $-3x^4$ by $3x^4$. We therefore find that the next term is -1.

As before, we multiply $3a^2 + 3ab + b^2$ by b, or $3x^4 - 6x^3 + 3x + 1$ by -1, thus completing the cube of $x^2 - x - 1$.

4. Find the cube root of $x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$. Leaving the divisors to be found by the student, we have

$$\begin{array}{l|l}
x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1 & x^2 + 2x + 1 \\
x^6 & \\
\hline
\quad 6x^5 + 15x^4 + 20x^3 & \\
\quad 6x^5 + 12x^4 + 8x^3 & \\
\hline
\qquad\quad 3x^4 + 12x^3 + 15x^2 + 6x + 1 & \\
\qquad\quad 3x^4 + 12x^3 + 15x^2 + 6x + 1 & \\
\hline
\end{array}$$

Exercise 176. Cube Root of Polynomials

Examples 1 to 4, oral — Examples 5 to 21, written

1. What is the cube root of $x^3 + 3x^2 + 3x + 1$?

2. What is the cube root of $x^3 + 6x^2 + 12x + 8$?

3. What is the cube root of $27 - 27x + 9x^2 - x^3$?

4. What are the first and the last terms of the cube root of the perfect cube $125x^9 + 300x^8 + 465x^7 + 424x^6 + 279x^5 + 108x^4 + 27x^3$?

Find the cube roots of the following polynomials:

5. $27x^3 - 54x^2y + 36xy^2 - 8y^3$.

6. $m^6 - 3m^4n + 3m^2n^2 - n^3$.

7. $125a^3 + 225a^2b + 135ab^2 + 27b^3$.

8. $343n^3 + 735n^2m + 125m^3 + 525m^2n$.

9. $x^6 - 3x^5y + 6x^4y^2 - 7x^3y^3 + 6x^2y^4 - 3xy^5 + y^6$.

10. $8a^6 - 60a^5 + 114a^4 + 55a^3 - 171a^2 - 135a - 27$.

11. $64x^6 - 144x^5y + 60x^4y^2 - y^6 - 9xy^5 - 15x^2y^4 + 45x^3y^3$.

12. $27n^6 + 189n^5 + 198n^4 - 791n^3 - 594n^2 + 1701n - 729$.

13. $x^{-3} + 9x^{-2}y + 27x^{-1}y^2 + 27y^3$.

14. $64m^{-3} - 144m^{-2}n^{-1} + 108m^{-1}n^{-2} - 27n^{-3}$.

15. $27x^{-9} - 135x^{-8} + 198x^{-7} - 35x^{-6} - 66x^{-5} - 15x^{-4} - x^{-3}$.

16. $27x^{\frac{6}{5}} + 108x + 198x^{\frac{4}{5}} + 208x^{\frac{3}{5}} + 132x^{\frac{2}{5}} + 48x^{\frac{1}{5}} + 8$.

17. $8x^{-\frac{6}{7}} - 12x^{-\frac{5}{7}} + 30x^{-\frac{4}{7}} - 25x^{-\frac{3}{7}} + 30x^{-\frac{2}{7}} - 12x^{-\frac{1}{7}} + 8$.

18. $\dfrac{8a^3}{27x^3} + \dfrac{4a^2}{3x^2} + \dfrac{3a}{x} + 4 + \dfrac{27x}{8a} + \dfrac{27x^2}{16a^2} + \dfrac{27x^3}{64a^3}$.

19. $8a^3 + 36a^2b - 12a^2c + 27b^3 + 54ab^2 + 6ac^2 - 27b^2c + 9bc^2 - c^3 - 36abc$.

20. $64x^{-12} - 144x^{-10} + 96x^{-9} + 108x^{-8} - 144x^{-7} + 21x^{-6} + 54x^{-5} - 36x^{-4} + 8x^{-3}$.

21. Write out a rule for cube root similar to that for square root on page 270.

292. Cube Root of Numbers. The first step in extracting the cube root of a number is to separate the figures of the number into groups of three figures each, called *periods.*

Since $1 = 1^3$, $1000 = 10^3$, $1,000,000 = 100^3$, and so on, the cube root of any integral number that has *one*, *two*, or *three* figures, is a number of *one* figure; the cube root of any integral number that has *four*, *five*, or *six* figures, is a number of *two* figures; and so on.

If, therefore, an integral number is separated into periods of three figures each, from right to left, the number of figures in the root will be equal to the number of periods. The last period to the left may consist of one figure, two figures, or three figures.

For example, the cube root of 34,645,976 will have three integral places.

Cube roots are required in practical work in engineering, but such roots are usually found by tables, logarithms, or some such mechanical devices as the slide rule or the computing machine. In order to use the computing machine as much knowledge of the theory is needed as is here set forth.

Find the cube root of 42,875.

We first recall that if $t =$ tens and $u =$ units, we have $(t + u)^3 = t^3 + 3t^2u + 3tu^2 + u^3$.

We see that there will be two integral places in the root. The first period, 42, contains t^3. The greatest cube in 42 is 27, and the cube root of 27 is 3. Hence $t = 3$.

$$\begin{array}{rr|l}
 & \multicolumn{2}{l}{42\ 875(35} \\
 & \multicolumn{2}{l}{27} \\
\cline{3-3}
3 \times 30^2 = & 2700 & 15\ 875 \\
3 \times (30 \times 5) = & 450 & \\
5^2 = & \underline{25} & \\
 & 3175 & 15\ 875 \\
\cline{3-3}
\end{array}$$

The remainder, 15,875, resulting from subtracting the cube of the tens, will contain $3t^2u + 3tu^2 + u^3$.

Each of these three parts contains u as a factor.

Hence the 15,875 consists of two factors, one of which is u, and the other is $3t^2 + 3tu + u^2$. The largest part of this second factor is $3t^2$.

If the 158 hundreds of the remainder is divided by $3t^2 = 3 \times 30^2$, or 27 hundreds, the quotient will be approximately u. The second factor can now be completed by adding to the 2700 the sum of $3 \times (30 \times 5)$, or 450, and 5^2, or 25.

If this factor, 3175, is now multiplied by 5, the result is 15,875, which completes the cube of 35. There being no remainder, $\sqrt[3]{42875} = 35$.

To check the work, $35^3 = 42,875$.

293. Cube Roots of Larger Numbers. The method of § 292 will apply to numbers of more than two periods, by considering *the part of the root already found as so many tens* with respect to the next figure of the root.

For example, find the cube root of 57,512,456.

		57 512 456(386
		27
$3 \times 30^2 =$	2700	30 512
$3 \times (30 \times 8) =$	720	
$8^2 =$	64	
	3484	27 872
$3 \times 380^2 =$	433200	2 640 456
$3 \times (380 \times 6) =$	6840	
$6^2 =$	36	
	440076	2 640 456

Therefore the cube root is 386.

294. Cube Roots of Decimals. If a cube root has decimal places, the cube will have *three times* as many.

Thus, if 0.11 is the cube root of a number, the number is $0.11 \times 0.11 \times 0.11 = 0.001331$. Hence, if a given number contains a decimal, we separate it into periods of three figures each, beginning at the decimal point and proceeding toward the left for the integral part, and toward the right for the decimal. The last period of the decimal must contain *three* figures, zeros being annexed when necessary.

Find the cube root of 187.149248.

		187.149 248(5.72
		125
$3 \times 50^2 =$	7500	62 149
$3 \times (50 \times 7) =$	1050	
$7^2 =$	49	
	8599	60 193
$3 \times 570^2 =$	974700	1 956 248
$3 \times (570 \times 2) =$	3420	
$2^2 =$	4	
	978124	1 956 248

Since there can be only one integral place, the decimal point is placed after the 5, the cube root being 5.72.

295. Approximate Cube Roots. If the given number is not a perfect cube, zeros may be annexed and a value of the root may be found as near to the true value as we please.

Extract the cube root of 1250.6894.

$$\begin{array}{rr|l}
 & & 1\ 250.689\ 400(10.77 \\
 & & 1 \\ \cline{3-3}
3 \times 10^2 = & 300 & 250
\end{array}$$

Since 300 is not contained in 250, the next figure of the root is 0.

$$\begin{array}{rr|l}
3 \times 100^2 = & 30000 & 250\ 689 \\
3 \times (100 \times 7) = & 2100 & \\
7^2 = & 49 & \\ \cline{2-2}
 & 32149 & 225\ 043 \\ \hline
3 \times 1070^2 = & 3434700 & 25\ 646\ 400 \\
3 \times (1070 \times 7) = & 22470 & \\
7^2 = & 49 & \\ \cline{2-2}
 & 3457219 & 24\ 200\ 533 \\ \cline{3-3}
 & & 1\ 445\ 867
\end{array}$$

Exercise 177. Cube Root of Numbers

Extract the cube roots of the following numbers:

1. 1,771,561.
2. 1,295,029.
3. 2,048,383.
4. 47,832,147.
5. 11,390,625.
6. 87,528,384.
7. 4826.809.
8. 0.000912673.
9. 0.114791256.

Find the edges of the cubes having the following volumes:

10. 75 cu. in. 11. 830 cu. ft. 12. 92.5 cu. in. 13. $7\frac{1}{2}$ cu. in.

14. Find the diameter of an iron ball that weighs 27 times as much as an iron ball 2 in. in diameter.

15. The weights of two iron cylinders of the same shape are as 2197 to 4913. Find the ratio of their heights.

16. Find the edge of a cube whose volume is equal to the volume of a rectangular solid $81'' \times 3'' \times 3''$.

17. Find the edge of a cubical cistern that holds as many gallons as a rectangular cistern $12' \times 8' \times 5\frac{1}{3}'$.

CHAPTER XX

PROGRESSIONS

296. Series. A succession of terms that proceed according to some fixed law is called a *series*.

For example, the natural numbers, 1, 2, 3, ⋯, form a series, the law being that each term is one more than the preceding term.

The study of series is an important part of higher mathematics. For example, by means of series we find the approximate value of π by an easier method than the one given in geometry.

297. Finite Series. A series in which the number of terms is limited is called a *finite series*.

If the number of terms of a series is unlimited it is called an *infinite series*.

For example, 2, 6, 18 is a finite series, having only three terms. The fixed law is that each term after the first is three times the preceding term of the series.

The series 1, 2, 3, ⋯ and so on forever, is an infinite series, as is also the series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$.

The number of different kinds of series is evidently unlimited. For example, $1^1, 2^2, 3^3, 4^4, \cdots$ is a series, and so are $1, -2, +3, -4, +5, -6, \cdots$, and $3^{-1}, 4^{+2}, 5^{-3}, 6^{+4}, \cdots$.

Only two kinds of series are commonly considered in elementary algebra, the arithmetical and the geometric.

298. Arithmetical Progression. A finite series in which each succeeding term after the first may be found by adding a constant quantity to the preceding term, is called an *arithmetical progression*.

The words *series* and *progression* are generally used interchangeably in elementary algebra.

For example, 2, 4, 6, 8, 10 is an arithmetical series or an arithmetical progression.

299. Elements of an Arithmetical Progression. An arithmetical progression may be represented by

$$a, \quad a+d, \quad a+2d, \quad a+3d, \cdots,$$

in which a is the first term and d is the constant quantity added.

It is customary to speak of d as the *common difference*, and to represent the various elements as follows:

$a =$ first term, $d =$ common difference,
$l =$ last term, $n =$ number of terms,

and to let s stand for the sum of all the terms.

If d is positive the series is an *increasing series*, as in the case of 2, 5, 8, 11, where $a = 2$, $d = 3$, $n = 4$, $l = 11$.

If d is negative the series is a *decreasing series*, as in the case of 16, 12, 8, 4, 0, -4, in which $a = 16$, $d = -4$, $n = 6$, $l = -4$.

The terms between the first and last terms are called *arithmetical means.*

300. The *n*th Term, or *l*. Since each succeeding term of an arithmetical progression is obtained by adding d to the preceding term, the coefficient of d is always one less than the number of the term. Hence the coefficient of d in the nth term is $(n-1)$ Calling the nth term l, we have

$$\boldsymbol{l = a + (n-1)d.}$$

Thus in the series 2, 5, 8, 11, we see that the last term is 11, and that $11 = 2 + (4-1) \cdot 3 = 2 + 3 \cdot 3$.

301. The Sum of the Terms. Indicating the sum of the terms, we have

$$\begin{aligned} & s = a + (a+d) + (a+2d) + \cdots + (l-d) + l \\ \text{or} \quad & \underline{s = l + (l-d) + (l-2d) + \cdots + (a+d) + a} \\ \therefore\ & 2s = (a+l) + (a+l) + (a+l) + \cdots + (a+l) + (a+l) \\ & \quad = n(a+l). \\ \therefore\ & s = \frac{n}{2}(a+l). \end{aligned}$$

That is, the formula for the sum is

$$\boldsymbol{s = \frac{n}{2}(a+l).}$$

302. Problems in Arithmetical Progressions. In the two formulas in §§ 300 and 301, when the values of any three of the letters are known, the values of the others may be found.

1. Find the tenth term in the series 1, 7, 13, ⋯.

We have $a = 1$, $d = 6$, $n = 10$.

Hence $$l = a + (n - 1)d = 1 + 9 \cdot 6 = 55.$$

2. Find the sum of the terms of the series 1, 7, 13, ⋯ to ten terms.

As in Ex. 1, $l = 55$.

Hence $$s = \frac{n}{2}(a + l) = \frac{10}{2}(1 + 55) = 5 \cdot 56 = 280.$$

3. Write the series of which the first term is 5, the last term 33, and the sum of the terms 152.

Since $l = a + (n - 1)d$, $\quad 33 = 5 + (n - 1)d.$

Since $s = \frac{n}{2}(a + l)$, $\quad 152 = \frac{n}{2}(5 + 33) = 19n.$

$$\therefore 8 = n.$$

$$\therefore 33 = 5 + (8 - 1)d,$$

or $$4 = d.$$

Therefore the series is 5, 9, 13, 17, 21, 25, 29, 33.

4. Find a when d, l, and s are given.

From $l = a + (n - 1)d$ we have $n = \dfrac{l - a + d}{d}$.

From $s = \frac{n}{2}(a + l)$ we have $\quad n = \dfrac{2s}{a + l}$.

Therefore $$\frac{l - a + d}{d} = \frac{2s}{a + l}.$$

Simplifying, $$a^2 - ad = l^2 + ld - 2ds.$$

Solving for a, $$a = \tfrac{1}{2}\left[d \pm \sqrt{(2l + d)^2 - 8ds}\right].$$

303. Formulas of Arithmetical Progressions. Only the two fundamental formulas of §§ 300 and 301 need to be memorized. From them the formulas on page 368 may be deduced.

Teachers will use their discretion as to the amount of this work to be required.

No.	Given	Required	Result
1	$a\ d\ n$	l	$l = a + (n-1)d$
2	$a\ d\ s$		$l = \frac{1}{2}\left[-d \pm \sqrt{8ds + (2a-d)^2}\right]$
3	$a\ n\ s$		$l = \frac{2s}{n} - a$
4	$d\ n\ s$		$l = \frac{s}{n} + \frac{(n-1)d}{2}$
5	$a\ d\ n$	s	$s = \frac{1}{2}n[2a + (n-1)d]$
6	$a\ d\ l$		$s = \frac{l+a}{2} + \frac{l^2 - a^2}{2d}$
7	$a\ n\ l$		$s = \frac{n}{2}(a + l)$
8	$d\ n\ l$		$s = \frac{1}{2}n[2l - (n-1)d]$
9	$d\ n\ l$	a	$a = l - (n-1)d$
10	$d\ n\ s$		$a = \frac{s}{n} - \frac{(n-1)d}{2}$
11	$d\ l\ s$		$a = \frac{1}{2}\left[d \pm \sqrt{(2l+d)^2 - 8ds}\right]$
12	$n\ l\ s$		$a = \frac{2s}{n} - l$
13	$a\ n\ l$	d	$d = \frac{l-a}{n-1}$
14	$a\ n\ s$		$d = \frac{2(s-an)}{n(n-1)}$
15	$a\ l\ s$		$d = \frac{l^2 - a^2}{2s - l - a}$
16	$n\ l\ s$		$d = \frac{2(nl - s)}{n(n-1)}$
17	$a\ d\ l$	n	$n = \frac{l-a}{d} + 1$
18	$a\ d\ s$		$n = \frac{d - 2a \pm \sqrt{(2a-d)^2 + 8ds}}{2d}$
19	$a\ l\ s$		$n = \frac{2s}{l+a}$
20	$d\ l\ s$		$n = \frac{2l + d \pm \sqrt{(2l+d)^2 - 8ds}}{2d}$

Exercise 178. Arithmetical Progression

All examples written

1. Find the ninth and twelfth terms of 3, 7, 11, ⋯.

2. Find the tenth and twentieth terms of 5, 11, 17, ⋯.

3. Find the sixth and eighth terms of 40, 32, 24, ⋯.

4. Find the fifteenth term of 12, 3, -6, ⋯.

5. Find the sum of the first nine terms of 7, 11, 15, ⋯; also of the first twelve terms.

6. Find the sum of the first eight terms of -28, -20, -12, ⋯; also of the first three terms; also of the first five terms.

7. Find the sum of the first seven terms of -6, -3, 0, ⋯; also of the first five terms.

8. Find the sum of the first sixteen terms of $3\frac{1}{2}$, 7, $10\frac{1}{2}$, ⋯; also of the first twenty terms.

9. Given $a = 7$, $l = 42$, $n = 6$, find d and s.

10. Given $a = 65$, $n = 9$, $s = 333$, find d and l.

11. Given $a = -27$, $d = 12$, $l = 45$, find n and s.

12. Given $a = 7$, $l = 49$, $s = 812$, find d and n.

13. Given $d = \frac{2}{3}$, $n = 24$, $s = 56$, find a and l.

14. Given $a = 4$, $d = 3$, $s = 246$, find l and n.

15. Insert four terms in an arithmetical series between -15 and 45, that is, four arithmetical means, thus making six terms in the series.

16. Insert seven arithmetical means between 3 and 51.

17. Insert twelve arithmetical means between 3 and 42.

18. Insert six arithmetical means between 1 and 5.

19. The first term of an arithmetical series is 21 and the third term is 33. Find the sum of five terms.

20. The first term of an arithmetical series is -3, and the sum of the first five terms is 105. What term is 33?

21. The sum of three numbers in an arithmetical series is 120. The difference between the first and last terms is 26. Find the series.

22. What is the sum of the first hundred positive integers?

23. What is the sum of the first ten odd numbers? of the first twenty? of the first n?

24. What is the sum of the first ten numbers that are divisible by 5?

25. What is the sum of the first ten numbers beginning with 15 that are divisible by 3?

26. In a potato race 100 potatoes are placed 3 ft. apart in a straight line. A runner picks up one potato at a time and carries it to a basket in the line of the potatoes, and 3 ft. back of the first one. How far does he run?

27. In a potato race, if there are 50 potatoes 6 ft. apart, and the basket is 6 ft. back of the first one, how far does the contestant run?

28. A body falling freely falls 16.08 ft. in the first second, and in each succeeding second 32.16 ft. more than in the second immediately preceding. If a stone dropped from a stationary balloon reaches the ground in 12 sec., how far does it fall in the last second? How high is the balloon?

29. A baseball was dropped from the top of Washington Monument, 550 ft. high, and was caught by an American League catcher. How fast was the ball falling when caught?

30. Some railroads use 24-hour time, the hours being numbered from 1 to 24. If a clock should strike the hours on this plan, how many strokes would it strike in one day?

31. How many terms of the series 18, 15, 12 must be taken to have their sum 60? Write the series and explain the double answer.

32. How many terms of the series 40, 30, 20 must be taken to have their sum 90? Explain the two answers.

304. Geometric Progression. A finite series in which each succeeding term may be found by multiplying the preceding term by a constant multiplier is called a *geometric progression.*

The constant multiplier is called the *ratio.*

The first term is designated by a, the last term by l, the ratio by r, the number of terms by n, and the sum by s.

Special examples of a geometric progression are

$$2,\ 4,\ 8,\ 16,\ 32,\ 64,\ 128, \cdots,$$

and

$$729,\ 243,\ 81,\ 27,\ 9,\ 3,\ 1,\ \tfrac{1}{3}, \cdots,$$

and the general form is

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4, \cdots.$$

The terms between the first and last terms are called *geometric means.*

305. The nth Term, or l. Since each succeeding term of a geometric progression, after the first, is obtained by multiplying the preceding term by r, the exponent is always one less than the number of the term, so that the second term is ar, the third is ar^2, the tenth is ar^9, and the nth is ar^{n-1}. Hence

$$l = ar^{n-1}.$$

Thus in the series 5, 15, 45, 135 we see that $135 = 5 \cdot 3^3 = 5 \cdot 27$.

306. The Sum of the Terms. To find the sum, we have

$$s = a + ar + ar^2 + \cdots + ar^{n-1}.$$

Multiplying by r, $rs = ar + ar^2 + \cdots + ar^{n-1} + ar^n$.

Subtracting, $rs - s = ar^n - a$, or $(r-1)s = ar^n - a$.

Hence $$s = \frac{ar^n - a}{r-1}.$$

Since $l = ar^{n-1}$, therefore $lr = ar^n$.

$$\therefore s = \frac{lr - a}{r-1}.$$

We therefore have two convenient formulas for s:

$$s = \frac{ar^n - a}{r-1},$$

and

$$s = \frac{lr - a}{r-1}.$$

307. Problems in Geometric Progressions. From the formulas in §§ 305 and 306, when the values of any three letters are known the values of the other two may be found.

In the case of finding n, however, logarithms are needed except when n is easily determined by inspection. This case is therefore considered in only one simple example at this time.

1. The sum of the terms of a geometric progression is 381, the first term is 3, and the last term 192. Find the ratio and the number of terms.

Since $l = ar^{n-1}$,

therefore $192 = 3r^{n-1}$.

Since $s = \dfrac{lr - a}{r - 1}$,

therefore $381 = \dfrac{192r - 3}{r - 1}$,

from which $r = 2$.

Substituting, $192 = 3 \cdot 2^{n-1}$,

and $64 = 2^{n-1}$.

Since $64 = 2^6$, $n - 1 = 6$, and $n = 7$.

2. Given r, n, s, find l.

From $l = ar^{n-1}$, $a = \dfrac{l}{r^{n-1}}$.

Substituting in $s = \dfrac{lr - a}{r - 1}$,

$$s = \frac{rl - \dfrac{l}{r^{n-1}}}{r - 1}$$

$$= \frac{l(r^n - 1)}{r^{n-1}(r - 1)}.$$

Solving for l, $l = \dfrac{sr^{n-1}(r - 1)}{r^n - 1}$.

3. Insert a geometric mean between 7 and 63.

We have $\dfrac{x}{7} = \dfrac{63}{x}$,

whence $x^2 = 7 \cdot 63 = 441$.

$$\therefore x = \pm\sqrt{441} = \pm 21.$$

Therefore the series is either 7, 21, 63, or 7, − 21, 63, r being 3 in the first case and − 3 in the second case.

[illegible] Progressions. [illegible] the three fun- [illegible] memorized. [illegible] using their [illegible]

[illegible]

$\ldots^{-1} = 0$

[illegible]

$\ldots^{-1} = 0$

[illegible]

$\ldots = 0$

Exercise 179. Problems in Geometric Progressions

All examples written

1. Find the fifth term of 3, 6, 12, $\cdots$; the tenth term.

2. Find the sixth term of 4, 12, 36, $\cdots$; the twelfth term.

3. Find the eighth term of 32, 16, 8, $\cdots$; the tenth term.

4. Find the ninth term of 4, -16, 64, $\cdots$; the tenth term.

5. Find the tenth term of 1, $\frac{1}{2}$, $\frac{1}{4}$, $\cdots$; of 1, 2, 4, $\cdots$.

6. Find the twelfth term of -3, 9, -27 $\cdots$; the thirteenth term; the twentieth term.

Find the sum of the following series:

7. 3, 6, 12, $\cdots$ to five terms; to ten terms.

8. 4, 12, 36, $\cdots$ to six terms; to twelve terms.

9. 2, -4, 8, $\cdots$ to five terms; to six terms.

10. 3, -9, 27 $\cdots$ to eight terms; to nine terms.

11. 1, $\frac{1}{2}$, $\frac{1}{4}$, $\cdots$ to eight terms; to fifteen terms.

12. Insert a geometric mean between 4 and 25 and find the sum of the three terms.

13. A geometric series is 3, $3r$, $3r^2$, 1029. Find r, and thus insert two geometric means between 3 and 1029. (Extract the root by factoring.)

14. Insert two geometric means between 2 and 1458.

15. Insert three geometric means between 2 and 32.

16. Given $a = 8$, $r = 2$, $s = 248$, find l and write the series.

17. Given $a = 32$, $r = \frac{1}{2}$, $n = 6$, find l and s.

18. Given $l = 3$, $r = \frac{1}{7}$, $n = 5$, find a and s.

19. Given $s = -42$, $a = -64$, $r = -\frac{1}{2}$, find l and write the series. How many terms are there?

20. Given $r = 6$, $n = 5$, $l = 1296$, find a and s.

21. The first term of a geometric series is 5 and the ratio is 2; what term is 1280?

309. Infinite Geometric Series. When r is a proper fraction the successive terms become numerically smaller and smaller. By taking n large enough we can therefore make the nth term as small as we please, bringing it nearer and nearer to zero.

Thus in the series $4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$, the terms are getting smaller and smaller. If we take $n = 14$, we shall have $l = \frac{1}{2048}$, a small fraction; and if we take $n = 21$, we shall have $l = \frac{1}{262144}$, a very small fraction.

Since $r < 1$, we may avoid negative terms by writing $s = \dfrac{ar^n - a}{r - 1}$ in the form $\dfrac{a - ar^n}{1 - r}$; and by taking n large enough we can make ar^n as near zero as we please, and can make s approach as near as we please to $\dfrac{a - 0}{1 - r}$, or $\dfrac{a}{1 - r}$. That is, $\dfrac{a}{1 - r}$ is the *limit* to which s approaches when r is a proper fraction.

For convenience it is usually said that $s = \dfrac{a}{1 - r}$, when r is a proper fraction and n is infinite.

The full statement is that the limit of s is $\dfrac{a}{1 - r}$, when r is a proper fraction and the number of terms increases without limit.

However far we extend the series, s lacks a little of being $\dfrac{a}{1 - r}$, but the further we extend it the nearer s approaches this limit, the difference between s and $\dfrac{a}{1 - r}$ being less then any assigned positive quantity.

1. Find the sum of the infinite series $10, 5, 2\frac{1}{2}, 1\frac{1}{4}, \cdots$.

Since $s = \dfrac{a}{1 - r}$, we have $s = \dfrac{10}{1 - \frac{1}{2}} = 20$. That is, the further we go in summing the series, the nearer the sum approaches 20.

2. Find the sum of the infinite series $9, -3, 1, -\frac{1}{3}, \cdots$.

Here $\quad a = 9$, and $r = -\frac{1}{3}$.

Therefore

$$s = \frac{a}{1 - r} = \frac{9}{1 - (-\frac{1}{3})} = \frac{9}{\frac{4}{3}} = \frac{27}{4} = 6\tfrac{3}{4}.$$

3. Find the sum of the infinite series $\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \cdots$.

Here $\quad a = \frac{1}{10}$, and $r = \frac{1}{10}$.

Therefore

$$s = \frac{a}{1 - r} = \frac{0.1}{1 - 0.1} = \frac{0.1}{0.9} = \frac{1}{9}.$$

That is, the limit of the decimal fraction $0.111 \cdots$ is $\frac{1}{9}$.

4. Find the value of the fraction 0.72232323 ⋯.

Such a fraction is called a *recurring decimal* (or *repeating* or *circulating* decimal). It may be written

$$0.72 + \tfrac{23}{10000} + \tfrac{23}{1000000} + \cdots$$

in which 0.72 is not part of the infinite geometric series that follows.

In the series, $a = \frac{23}{10000}$ and $r = \frac{1}{100}$. Hence

$$s = \frac{a}{1-r} = \frac{\frac{23}{10000}}{1-\frac{1}{100}} = \frac{23}{9900}.$$

Add 0.72, the part of the decimal that does not recur, and

$$0.72 + \tfrac{23}{9900} = \tfrac{7151}{9900}.$$

If we reduce $\frac{7151}{9900}$ to a decimal fraction, we shall find that it equals 0.722323 ⋯, thus checking the work.

Exercise 180. Infinite Geometric Series

All examples written

1. Find the sum of the infinite series 15, 5, $1\frac{2}{3}$, ⋯.
2. Find the sum of the infinite series 12, 3, $\frac{3}{4}$, ⋯.
3. Find the sum of the infinite series 32, − 16, 8, − 4, ⋯.

Find the value of the following recurring decimals:

4. 0.2727 ⋯.
5. 0.3030 ⋯.
6. 0.481481 ⋯.
7. 0.520520 ⋯.
8. 0.76565 ⋯.
9. 0.83421421 ⋯.

Find the sum of the following infinite series:

10. 100, 50, 25, ⋯.
11. 99, 33, 11, ⋯.
12. 160, 40, 10, ⋯.
13. 625, 125, 25, ⋯.
14. 111, 74, $49\frac{1}{3}$, ⋯.
15. 625, 250, 100, ⋯.
16. 1250, 750, 450, ⋯.
17. 4096, 3584, 3136, ⋯.

18. If you take half of a line 4 in. long, and half of what is left, and so on, what is the limit of the sum of these parts?

19. If it were possible for a rubber ball to fall 10 ft. and bound back 5 ft., then to fall 5 ft. and bound back $2\frac{1}{2}$ ft., and to continue this forever, what is the limit of the distance through which the ball would pass?

CHAPTER XXI

THE BINOMIAL THEOREM

310. Factorial. The product of the positive integers from one to any given number n, inclusive, is called *factorial n*.

That is, $1 \cdot 2 \cdot 3 \cdot 4 =$ factorial 4, and $1 \cdot 2 \cdot 3 \cdots n =$ factorial n.

There are two common symbols for factorial n, as follows: $\underline{|n}$ and $n\,!$, the former being more convenient to write and the latter more convenient to print. Thus $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = \underline{|5} = 5\,! = 120$.

311. Binomial Theorem. In order to develop the Binomial Theorem (§ 206) we may write the expansion of the first three powers of $a + b$ in the following form:

$$(a+b)^1 = a + b;$$

$$(a+b)^2 = a^2 + 2\,ab + \frac{2 \cdot 1}{2\,!}\,b^2;$$

$$(a+b)^3 = a^3 + 3\,a^2b + \frac{3 \cdot 2}{2\,!}\,ab^2 + \frac{3 \cdot 2 \cdot 1}{3\,!}\,b^3.$$

We therefore infer that, in the case of $(a+b)^n$,

1. *The number of terms is greater by one than the exponent of the power to which the binomial is raised.*

2. *The exponent of a in the first term is n, and it decreases by one to the right.*

3. *The exponent of b in the first term is 0, and it increases by one to the right.*

4. *The coefficient of the first term is 1, and of the second term n.*

5. *The coefficient of each term after the first is found from the next preceding term by multiplying the coefficient of that term by the exponent of a and dividing the product by a number greater by one than the exponent of b.*

312. Proof of the Binomial Theorem. We know that the laws just stated hold for the third power, because we can obtain this power (§ 311) by actual multiplication.

Let us, for the moment, assume that they hold for the *k*th power, *k* being any positive integer. Then

$$(a+b)^k = a^k + ka^{k-1}b + \frac{k(k-1)}{2!}a^{k-2}b^2 + \frac{k(k-1)(k-2)}{3!}a^{k-3}b^3 + \cdots. \quad (1)$$

If we multiply both members of (1) by $a+b$ in the usual manner, we have

$$(a+b)^{k+1} = a^{k+1} + (k+1)a^kb + \frac{(k+1)k}{2!}a^{k-1}b^2 + \frac{(k+1)k(k-1)}{3!}a^{k-2}b^2 + \cdots. \quad (2)$$

But (2) is exactly what we obtain if we expand $(a+b)^{k+1}$ by the Binomial Theorem.

Therefore, *if* the law holds for the *k*th power (that is, if (1) is true), it holds for the $(k+1)$th power, for we have shown this by actually multiplying (1) by $a+b$.

But the law *does* hold true for the *third* power (§ 311), and therefore it must hold true for the $(3+1)$th or *fourth* power. Since it holds true for the fourth power, it must hold true for the $(4+1)$th, or fifth power, and so on for any positive integral power. Therefore, for the *n*th power,

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots$$

This is a method of proof known as *mathematical induction.*

Evidently we may interchange a and b. This will give us the last few terms of the series, just as we have now the first few, thus:

$$\cdots + \frac{n(n-1)}{2!}a^2b^{n-2} + nab^{n-1} + b^n.$$

This is illustrated in such a familiar case as $a^3 + 3a^2b + 3ab^2 + b^3$.

1. Expand $(a-b)^n$.

Evidently the even powers of $-b$ are positive (§ 205) and the odd powers negative. We therefore have

$$(a-b)^n = a^n - na^{n-1}b + \frac{n(n-1)}{2\,!}a^{n-2}b^2 - \frac{n(n-1)(n-2)}{3\,!}a^{n-3}b^3 + \cdots.$$

We cannot tell the sign of the last term unless we know whether n is odd or even. It is, however, $(-b)^n$.

2. Expand $(1+x)^n$.

Substituting 1 for a, and x for b, in $(a+b)^n$, we have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2\,!}x^2 + \frac{n(n-1)(n-2)}{3\,!}x^3 + \cdots$$
$$+ \frac{n(n-1)}{2\,!}x^{n-2} + nx^{n-1} + x^n.$$

3. Expand $(1-x)^n$.

In Ex. 2 put $-x$ for x, and we have

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2\,!}x^2 - \frac{n(n-1)(n-2)}{3\,!}x^3 + \cdots.$$

We cannot tell the sign of the last term unless we know whether n is odd or even. It is, however, $(-x)^n$.

4. Expand $(1+2a)^6$.

Substituting 1 for a, and $2a$ for b, in $(a+b)^n$, we have

$$(1+2a)^6 = 1 + 6(2a) + \frac{6\cdot5}{2\,!}(2a)^2 + \frac{6\cdot5\cdot4}{3\,!}(2a)^3 + \frac{6\cdot5\cdot4\cdot3}{4\,!}(2a)^4$$
$$+ \frac{6\cdot5\cdot4\cdot3\cdot2}{5\,!}(2a)^5 + \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{6\,!}(2a)^6$$
$$= 1 + 12a + 60a^2 + 160a^3 + 240a^4 + 192a^5 + 64a^6.$$

5. Expand to four terms $\left(\frac{2}{x} - \frac{3x^2}{4}\right)^7$.

Substituting $\frac{2}{x}$ for a, and $-\frac{3x^2}{4}$ for b, in $(a+b)^n$, we have

$$\left(\frac{2}{x} - \frac{3x^2}{4}\right)^7 = \left(\frac{2}{x}\right)^7 - 7\left(\frac{2}{x}\right)^6\left(\frac{3x^2}{4}\right) + 21\left(\frac{2}{x}\right)^5\left(\frac{3x^2}{4}\right)^2 - 35\left(\frac{2}{x}\right)^4\left(\frac{3x^2}{4}\right)^3 + \cdots$$
$$= \frac{128}{x^7} - \frac{336}{x^4} + \frac{378}{x} - \frac{945x^2}{4} + \cdots.$$

We could also have written this $(2x^{-1} - \frac{3}{4}x^2)^7$ and expanded, obtaining $128x^{-7} - 336x^{-4} + 378x^{-1} - \frac{945}{4}x^2 + \cdots$.

313. Formula for the rth term. We see by § 311 that the third term is
$$\frac{n(n-1)}{2!}a^{n-2}b^2,$$
the fourth term is $\dfrac{n(n-1)(n-2)}{3!}a^{n-3}b^3, \ldots$

and hence that the rth term is
$$\frac{n(n-1)(n-2)\cdots\text{ to }(r-1)\text{ factors}}{(r-1)!}a^{n-(r-1)}b^{r-1}.$$

Find the eighth term of $\left(4-\dfrac{x^2}{2}\right)^{10}$.

Here $a=4,\ b=-\tfrac{1}{2}x^2,\ n=10,\ r=8.$

The eighth term is
$$\frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4}{7!}(4)^{10-7}(-\tfrac{1}{2}x^2)^7,\text{ or }-60x^{14}.$$

Exercise 181. Binomial Theorem

Examples 1 to 3, oral — Examples 4 to 20, written

1. Expand $(1+x)^3$; $(1-x)^3$; $(a+2)^3$; $(2-a)^3$.
2. Expand $(a+b)^4$; $(a-b)^4$; $(a+1)^4$; $(1-a)^4$.
3. Expand $(a+b)^5$; $(a+1)^5$; $(a-b)^5$; $(1-a)^5$.

Expand the following expressions:

4. $(a+2b)^4$.
5. $(a-2b)^5$.
6. $(\tfrac{1}{2}a+\tfrac{3}{4}b)^4$.
7. $\left(\dfrac{2}{x}-\dfrac{x}{2}\right)^5$.
8. $(a+2b)^6$.
9. $(2a-3b)^6$.
10. $(\tfrac{1}{2}a+\tfrac{2}{3}b)^6$.
11. $\left(\dfrac{3}{x}-\dfrac{x^2}{5}\right)^6$.
12. $(a^{-2}+b^{-3})^3$.
13. $(2a^{-3}-x^{-4})^6$.
14. $(\tfrac{1}{2}\sqrt{x}-\tfrac{2}{3}\sqrt{y})^3$.
15. $\left(\sqrt[3]{\dfrac{x^2}{3}}-\sqrt{\dfrac{2}{x}}\right)^3$.

16. In $(3x-4y)^8$ find the fifth term; the sixth.
17. In $(2x+3)^{12}$ find the ninth term; the tenth.
18. In $(3x-\tfrac{1}{2}y^2)^{18}$ find the tenth term; the twelfth.
19. In $(4a-3b^2)^{14}$ find the middle term.
20. In $(a^{\frac{1}{2}}+b^{\frac{1}{3}})^{10}$ find the middle term.

314. Convergent Series. A series in which the sum of the terms, as the number of terms is indefinitely increased, approaches some fixed finite value as a limit, is called a *convergent series.*

For example, as shown in § 309, when the number of terms is indefinitely increased the limit approached by $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ is $\frac{1}{1-\frac{1}{2}}$, or 2. This is therefore a convergent series.

But the series 2, 4, 6, $\cdots$ is not a convergent series, because, when the number of terms is indefinitely increased, $2 + 4 + 6 + \cdots$ becomes infinite. Such a series is called a *divergent series.*

315. Binomial Theorem, Any Exponent. The Binomial Theorem is true for any exponent, integral or fractional, positive or negative, provided the series that results is convergent.

The proof of this fact is not suited to elementary algebra, and it has to be assumed at this time.

1. Expand to four terms $(x + y)^{\frac{1}{2}}$.

Substituting in the expansion of $(a + b)^n$, we have

$$\begin{aligned}(x+y)^{\frac{1}{2}} &= x^{\frac{1}{2}} + \tfrac{1}{2}x^{\frac{1}{2}-1}y + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2\,!}x^{\frac{1}{2}-2}y^2 \\ &\quad + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3\,!}x^{\frac{1}{2}-3}y^3 + \cdots \\ &= x^{\frac{1}{2}} + \frac{y}{2x^{\frac{1}{2}}} - \frac{y^2}{8x^{\frac{3}{2}}} + \frac{y^3}{16x^{\frac{5}{2}}} - \cdots.\end{aligned}$$

2. Expand to three terms $\sqrt[3]{9}$.

$$\begin{aligned}\sqrt[3]{9} = \sqrt[3]{8+1} &= (8+1)^{\frac{1}{3}} \\ &= 8^{\frac{1}{3}} + \tfrac{1}{3}\cdot 8^{\frac{1}{3}-1} + \frac{\frac{1}{3}(\frac{1}{3}-1)}{2\,!}8^{\frac{1}{3}-2} + \cdots \\ &= 2 + \tfrac{1}{12} - \tfrac{1}{288} + \cdots = 2.079 + .\end{aligned}$$

3. Expand to three terms $(2x - \frac{1}{3}y)^{-\frac{2}{3}}$.

Substituting in the expansion of $(a + b)^n$, we have

$$\begin{aligned}(2x - \tfrac{1}{3}y)^{-\frac{2}{3}} &= (2x)^{-\frac{2}{3}} + (-\tfrac{2}{3})(2x)^{-\frac{2}{3}-1}(-\tfrac{1}{3}y) \\ &\quad + \frac{(-\frac{2}{3})(-\frac{2}{3}-1)}{2\,!}(2x)^{-\frac{2}{3}-2}(-\tfrac{1}{3}y)^2 + \cdots \\ &= (2x)^{-\frac{2}{3}} + \tfrac{2}{9}(2x)^{-\frac{5}{3}}y + \tfrac{5}{81}(2x)^{-\frac{8}{3}}y^2 + \cdots.\end{aligned}$$

4. Expand to four terms $(1-x)^{-1}$.

$$(1-x)^{-1}=1+(-1)\cdot 1^{-1-1}(-x)+\frac{(-1)(-1-1)}{2!}1^{-1-2}(-x)^2+\cdots$$
$$=1+x+x^2+x^3+\cdots.$$

As a special case, suppose $x=2$. We then have

$$(1-2)^{-1}=1+2+2^2+2^3+\cdots.$$

But $$(1-2)^{-1}=\frac{1}{1-2}=\frac{1}{-1}=-1.$$

Therefore it would seem that $-1=1+2+2^2+2^3+\cdots$, which is absurd. The trouble is easily seen, however, for the series $1+2+2^2+2^3+\cdots$ is manifestly not convergent, and the Binomial Theorem does not hold for such cases (§ 315).

Exercise 182. Binomial Theorem, any Exponent

All examples written

1. Expand to four terms: $(1+n)^{\frac{1}{2}}$; $(1+n)^{-2}$.

2. Expand $(1+x)^{-\frac{1}{2}}$ to four terms, and write the result with fractions and radical signs instead of negative and fractional exponents.

Expand the following expressions to four terms:

3. $(x-1)^{-3}$.
4. $(a-x)^{-4}$.
5. $(2a+1)^{-3}$.
6. $(1+x)^{\frac{2}{3}}$.
7. $(1-x)^{-\frac{2}{3}}$.
8. $(a-2x)^{-\frac{2}{3}}$.
9. $(a^2b^{-2}+1)^{\frac{1}{2}}$.
10. $(8a^{-3}-1)^{\frac{1}{3}}$.
11. $(8x^{-3}-27)^{-\frac{2}{3}}$.

12. In $(a-b)^{-5}$ find the fourth term.

13. In $(a-3)^{-6}$ find the fifth term; the sixth.

14. In $(a+x)^{\frac{1}{2}}$ find the third term; the fourth.

15. In $(2a-1)^{\frac{1}{3}}$ find the third term; the fifth.

16. Expand to three terms: $\sqrt{5}$, $\sqrt[3]{28}$, $\sqrt[5]{33}$.

17. In $\left(x-\dfrac{1}{x}\right)^{12}$, what is the term that does not contain x?

18. Find the square root of 10 to three decimal places by the Binomial Theorem. Verify the result by finding the square root by the ordinary method.

CHAPTER XXII

LOGARITHMS

316. Logarithm. The power to which a given number, called the *base*, must be raised to equal another number is called the *logarithm* of this other number.

For example, since $10^3 = 1000$,

therefore, to the base 10, $3 =$ the logarithm of 1000, written log 1000.

317. Base. We may take various bases for systems of logarithms, but for practical calculation 10 is taken.

Since $10^2 = 100$, and $10^3 = 1000$, we know that the logarithm of any number between 100 and 1000 must lie between 2 and 3. For example, we know that log 475 is 2 + some fraction.

Since $\log 10 = 1$, because $10^1 = 10$,

and $\log 1 = 0$, because $10^0 = 1$,

and $\log \frac{1}{10} = -1$, because $10^{-1} = \frac{1}{10}$,

we see that *the logarithm of the base is 1, the logarithm of 1 is zero, and the logarithm of a proper fraction is negative.*

Exercise 183. Logarithms

1. What is $\log \frac{1}{100}$, or log 0.01? log 0.001? log 0.0001?

Write the integers between which lie the logarithms of:

2. 83.
3. 83.5.
4. 127.
5. 127.96.
6. 4237.
7. 4237.8.
8. 42,756.
9. 42,756.95.

Show that the following statements are true:

10. $\log 1 + \log 10 + \log 100 + \log 1000 + \log 0.001 = 3.$

11. $7 \log 1 + 9 \log 10{,}000 + \log 0.1 + \log 0.000001 = 29.$

318. Logarithm of a Product. *The logarithm of the product of two factors equals the sum of the logarithms of the factors.*

Let A and B be the factors, and x and y their logarithms Then, remembering that $x = \log A$ and $y = \log B$, we have

$$A = 10^x,$$

and $$B = 10^y.$$

Hence $$AB = 10^{x+y},$$

and therefore $$\log AB = x + y$$
$$= \log A + \log B.$$

Such laws are easily proved for any base.

319. Logarithm of a Quotient. *The logarithm of the quotient of two numbers equals the logarithm of the dividend minus the logarithm of the divisor.*

For if $x = \log A$, then $A = 10^x$,

and if $y = \log B$, then $B = 10^y$.

Hence $$\frac{A}{B} = 10^{x-y},$$

and therefore $$\log \frac{A}{B} = x - y$$
$$= \log A - \log B.$$

It is seen that if we know the logarithms of all numbers, we can find the logarithm of a product by addition and the logarithm of a quotient by subtraction. If we can then find the numbers of which these results are the logarithms, we shall have solved our problems in multiplication and division by merely adding and subtracting.

320. Logarithm of a Power. *The logarithm of a power of a number equals the logarithm of the number multiplied by the exponent.*

If $x = \log A$, then $A = 10^x$.

Raising to the pth power, $A^p = 10^{px}$.

Hence $$\log A^p = px$$
$$= p \log A.$$

321. Logarithm of a Root. *The logarithm of a root of a number equals the logarithm of the number divided by the index of the root.*

If $x = \log A$, then $$A = 10^x.$$

Taking the rth root, $$A^{\frac{1}{r}} = 10^{\frac{x}{r}}.$$

Hence $$\log A^{\frac{1}{r}} = \frac{x}{r} = \frac{\log A}{r}.$$

Therefore the operations of multiplication, division, raising to powers, and extracting roots will be greatly simplified, as already stated in § 319, if we can find the logarithms of numbers.

322. Characteristic and Mantissa. Usually a logarithm consists of an integer plus a decimal fraction. The integral part of a logarithm is called the *characteristic.*

The decimal part of a logarithm is called the *mantissa.*

Thus if $\log 2353 = 3.3716$, the characteristic is 3 and the mantissa is 0.3716. This means that $10^{3.3716} = 2353$, approximately.

323. Finding the Characteristic. Since we know that

$$10^3 = 1000 \quad \text{and} \quad 10^4 = 10{,}000,$$

therefore $$3 = \log 1000 \quad \text{and} \quad 4 = \log 10{,}000.$$

Hence the logarithm of a number between 1000 and 10,000 lies between 3 and 4 and is therefore 3 plus some fraction.

Since $10^{-2} = 0.01$ and $10^{-3} = 0.001$,

therefore $-2 = \log 0.01$ and $-3 = \log 0.001$.

Hence the logarithm of a number between 0.001 and 0.01 lies between -3 and -2 and is -3 plus some fraction.

For convenience *the mantissa is always taken as positive,* but the characteristic may be either positive or negative.

Since we see that the characteristic is easily found without the aid of tables, the fact that it may be either positive or negative does not present any serious difficulties.

324. Rule for the Characteristic. From the reasoning set forth in § 323 we deduce the following rule:

1. *The characteristic of a number greater than 1 is positive, and is one less than the number of integral places in the number.*

For example, $\log 75 = 1 +$ some mantissa,
$\log 472.8 = 2 +$ some mantissa,
and $\log 14{,}800.75 = 4 +$ some mantissa.

2. *The characteristic of a number between 0 and 1 is negative, and is one more than the number of zeros between the decimal point and the first significant figure in the decimal.*

For example, $\log 0.03 = -2 +$ some mantissa,
and $\log 0.00076 = -4 +$ some mantissa.

The logarithm of a negative number is an imaginary number, and hence such logarithms are not used in computation.

325. The Negative Characteristic. If $\log 0.02 = -2 + 0.3010$, we cannot write it -2.3010, because this would mean that both mantissa and characteristic are negative. It is therefore written $\bar{2}.3010$, which means that only the characteristic 2 is negative.

That is, $\bar{2}.3010 = -2 + 0.3010$. We may also write it $0.3010 - 2$, or $8.3010 - 10$, or in any similar manner that will show that the characteristic is negative.

Exercise 184. Characteristics

Write the characteristics of the logarithms of:

1. 24.
2. 24.8.
3. 248.
4. 2.48.
5. 2480.
6. 2485.
7. 2485.7.
8. 2485.72.
9. 7235.
10. 723.5.
11. 72.35.
12. 7.235.
13. 72,350.
14. 0.7235.
15. 0.07235.
16. 0.007235.
17. 0.8.
18. 0.08.
19. 0.88.
20. 0.885.
21. 0.005.
22. 0.0051.
23. 0.0005.
24. 0.00051.
25. 0.0003.
26. 0.0033.
27. 0.0333.
28. 0.3333.
29. 0.0303.
30. 1.0303.
31. 2.0303.
32. 10.0303.

326. Mantissa independent of Decimal Point. It can be shown that $10^{3.3711} = 2350$, and therefore $\log 2350 = 3.3711$.

Dividing by 10 we have

$$10^{3.3711-1} = 235, \text{ and therefore } \log 235 = 2.3711.$$

Dividing each member of the first equation by 10^4, or 10,000, we have

$$10^{3.3711-4} = 0.235, \text{ and therefore } \log 0.235 = \bar{1}.3711.$$

That is, the mantissas are the same for log 2350, log 235, log 0.235, and so on, wherever the decimal point is placed.

The mantissa of the logarithm of a number is unchanged by any change in the position of the decimal point of the number.

This is a fact of great importance, for if the table of logarithms, which we shall soon describe, gives us the mantissa of log 235, we know that we may use the same mantissa for log 0.00235, log 2.35, and so on.

Exercise 185. Mantissas and Characteristics

Given log 625 = 2.7959, find:

1. log 62.5.
2. log 6.25.
3. log 0.625.
4. log 6250.
5. log 62,500.
6. log 0.0625.
7. log 625,000.
8. log 0.00625.
9. log 6,250,000.

Given log 16,630 = 4.2209, find:

10. log 1.663.
11. log 16.63.
12. log 166.3.
13. log 0.1663.
14. log 0.01663.
15. log 0.001663.
16. log 166,300.
17. log 1,663,000.
18. log 16,630,000.

Given log 9.154 = 0.9616, find:

19. log 91.54.
20. log 915.4.
21. log 0.9154.
22. log 0.09154.
23. log 9154.
24. log 915,400.

Given $\log \pi = 0.4971$, *find:*

25. $\log \pi^2$.
26. $\log \pi^3$.
27. $\log \pi^4$.
28. $\log \pi^{10}$.
29. $\log \pi^{\frac{1}{2}}$.
30. $\log \pi^{\frac{1}{3}}$.
31. $\log \sqrt[4]{\pi}$.
32. $\log \sqrt[5]{\pi}$.

327. Table of Logarithms. A table of logarithms to four decimal places is given on pages 390 and 391. It gives the mantissas for all integral numbers less than 1000, the decimal points in the mantissas being omitted.

Such a table is called a "four-place table."

Tables in which the mantissas are given to more than four decimal places are used when greater accuracy is required, but for ordinary computations a four-place table is usually sufficient to give results that are accurate to four significant figures.

In the table the numbers are given under N and the tenths under the columns headed 0, 1, 2, $\cdots$ 9.

Since only the mantissas are given, *always write the characteristic before looking up the mantissa, so that it shall not be forgotten.*

328. Finding the Logarithm of a Number. The following examples explain the use of the table:

1. Find the logarithm of 73.4.

First write the characteristic, 1.

In column N look for the first two figures, 73.

Then look to the right of 73 and in column 4. Here the mantissa is found to be 0.8657.

Hence $\log 73.4 = 1.8657$.

2. Find the logarithm of 4236.

First write the characteristic, 3.

Then notice that 4236 is between 4230 and 4240 and is 0.6 of the way from 4230 to 4240.

Hence we may assume that log 4236 is approximately 0.6 of the way from log 4230 to log 4240.

In column N look for the first two figures, 42.

Then look to the right of 42 and in columns 3 and 4. Here we find that

$$\begin{aligned} \log 4240 &= 3.6274 \\ \log 4230 &= \underline{3.6263} \\ 0.6 \text{ of } 0.0011 &= 0.0007, \text{ nearly.} \end{aligned}$$

Adding 0.0007 to 3.6263, we have $\log 4236 = 3.6270$.

This plan of finding the logarithm of a number of more significant figures than those given in the tables is called *interpolation*.

3. Find the logarithm of 0.0002705.

First write the characteristic, – 4.

Proceeding as in Ex. 2, we have

$$\begin{aligned} \log 0.000271 &= 0.4330 - 4 \\ \log 0.000270 &= \underline{0.4314 - 4} \\ 0.5 \text{ of } 0.0016 &= 0.0008 \end{aligned}$$

Adding 0.0008 to 0.4314 – 4, we have 0.4322 – 4.

We may write log 0.0002705 with the – 4 at the left, thus: $\bar{4}.4322$. When we have subtractions to perform, however, it is less confusing to place the negative characteristic at the right as shown above. It is also convenient to write the negative characteristic at the right in performing other operations on logarithms.

4. Find the logarithm of 7.

Since the mantissa of log 7 is the same as that of log 700, we look for 70 in column N and under column 0. Hence log 7 = 0.8451.

Exercise 186. Finding Logarithms

Using the table on pages 390 and 391, find the logarithms of:

1. 24.
2. 32.
3. 76.
4. 48.
5. 60.
6. 100.
7. 200.
8. 270.
9. 275.
10. 2756.
11. 27.56.
12. 275.60.
13. 27,560.
14. 32,450.
15. 41,270.
16. 4.
17. 7.
18. 0.4.
19. 0.7.
20. 0.44.
21. 0.77.
22. 1.44.
23. 1.77.
24. 17.7.
25. 177.
26. 1770.
27. 1775.
28. 17,750.
29. 25,300.
30. 25,350.
31. 22.
32. 222.
33. 222.2.
34. 2222.
35. 0.22.
36. 0.022.
37. 0.222.
38. 0.277.
39. 3.270.
40. 0.5000.
41. 0.5500.
42. 0.5550.
43. 0.5555.
44. 55,500.
45. 55,550.
46. 182.
47. 182.3.
48. 182.9.
49. 18.29.
50. 1.829.
51. 427.
52. 4275.
53. 42.75.
54. 4.275.
55. 427.5.
56. 42,750.
57. 53,750.
58. 50,000.
59. 50,050.
60. 75,080.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

329. Antilogarithm. The number corresponding to a given logarithm is called an *antilogarithm.*

Thus if log 676 is 2.8299, the antilogarithm of 2.8299 is 676.

330. Finding Antilogarithms. Antilogarithms are found from the table by looking for the number corresponding to the given mantissa, and locating the decimal point according to the characteristic.

1. Find the antilogarithm of 3.4265.

Looking for the mantissa 0.4265, we find that it is opposite 26 in column N and under column 7. It is therefore the mantissa of 267.

Since the characteristic is 3, there must be four integral places in the antilogarithm. Hence the antilogarithm must be 2670.

Therefore the antilogarithm of 3.4265 is 2670.

2. Find the antilogarithm of $\bar{2}.8404$.

Looking for the mantissa 0.8404, we do not find it in the table. We find that it lies between 0.8401 and 0.8407, their difference being 0.0006. The difference between 0.8401 and 0.8404 is 0.0003. Hence 0.8404 is $\frac{3}{6}$ of the way from 0.8401 to 0.8407.

Hence the antilogarithm of 0.8404 must be approximately $\frac{3}{6}$ of the way from the antilogarithm of 0.8401 to that of 0.8407.

We therefore write

$$\begin{aligned} \text{antilog}\, \bar{2}.8407 &= 0.06930 \\ \text{antilog}\, \bar{2}.8401 &= \underline{0.06920} \\ \tfrac{3}{6} \text{ (or } \tfrac{1}{2}\text{) of } & 0.00010 = 0.00005. \end{aligned}$$

Adding 0.00005 to 0.06920 we have

$$\text{antilog}\, \bar{2}.8404 = 0.06925.$$

3. Find the antilogarithm of 0.3664.

Looking for the mantissa 0.3664, we find that it lies between 0.3655 and 0.3674, whose difference is 0.0019. Since $0.3664 - 0.3655 = 0.0009$, the given mantissa is $\frac{9}{19}$ of the way from 0.3655 to 0.3664. But antilog $0.3655 = 2.32$, and antilog 0.3674 is 2.33. Adding $\frac{9}{19}$ of the difference to 2.32, we have 2.325, the antilogarithm required.

4. Find the antilogarithm of 7.9050.

The mantissa is evidently $\frac{3}{6}$ of the way from the mantissa for 803 and that for 804. Hence it is the mantissa for 8035. The characteristic being 7, the antilogarithm is 80,350,000.

Exercise 187. Antilogarithms

Using the table, find the antilogarithms of:

1. 0.4771.
2. 1.5682.
3. 3.8451.
4. $\bar{1}.8865$.
5. 0.5065.
6. 2.5211.
7. 4.5977.
8. 3.8785.
9. $\bar{2}.9380$.
10. $\bar{3}.9741$.
11. 0.1945.
12. 1.2266.
13. 2.8212.
14. $\bar{2}.8296$.
15. $\bar{4}.8398$.
16. 4.8397.
17. 1.8845.
18. 2.8844.
19. $\bar{2}.8846$.
20. 3.8851.
21. 0.0000.
22. 5.0000.
23. 2.7408.
24. 3.7406.
25. $\bar{2}.7410$.
26. 3.7735.
27. 2.2620.
28. 3.4210.
29. $\bar{1}.7280$.
30. $\bar{2}.6666$.
31. 0.7782 – 1.
32. 0.7864 – 2.
33. 0.7668 – 3.
34. 0.8028 – 4.
35. 9.8096 – 10.
36. 9.8235 – 10.
37. 8.8306 – 10.
38. 8.8500 – 10.
39. 8.8503 – 10.
40. 7.9996 – 10.

41. If the logarithm of the product of two numbers is 3.5211, what is the product of the numbers?

42. If the logarithm of the quotient of two numbers is 2.8370, what is the quotient of the numbers?

43. If the logarithm of the square of a certain number is 2.1584, what is the square of the number? What is the number?

44. If the logarithm of the square root of a certain number is 1.3979, what is the square root of the number? What is the number?

45. If we wish to multiply 277 by 49.8, what logarithms do we need? Find these logarithms from the table.

46. If we know that the logarithm of a certain result that we are seeking is 3.8293, what is the result?

47. There is a certain number such that the logarithm of its square is 3.8062. What is the number?

48. If the logarithm of the cube root of a certain number is 0.6551, what is the logarithm of the number? What is the cube root of the number?

331. Computation by Logarithms. Since we have learned (§§ 318–321) how to find the logarithms of products, quotients, powers, and roots, and (§ 330) how to find antilogarithms, we may now consider a few typical problems in computation.

1. Find the product of 247 and 3.95.

$$\begin{aligned} \log 247 &= 2.3927 \\ \log 3.95 &= \underline{0.5966} \\ & \quad 2.9893 = \log 975.8. \end{aligned}$$

If we multiply 247 by 3.95 the product is found to be 975.65, or 975.7, if we carry the result to only four significant figures. Our last figure is therefore 1 too great. This shows us that computations by logarithms give, in general, results that are only approximately correct. The approximation is closer when we use larger tables. In all work in this book results should be carried to four significant figures only, and it should be understood that the last figure may be incorrect.

2. Find the product of 37.2, 0.416, and -3.275.

Since we cannot find the logarithm of a negative number, we proceed to find the product of 37.2, 0.416, and 3.275, prefixing the negative sign to the result. This we have the right to do because the product of two positive quantities is positive, and when this product is multiplied by a negative number, the result is negative.

$$\begin{aligned} \log 37.2 &= 1.5705 \\ \log 0.416 &= 0.6191 - 1 \\ \log 3.275 &= \underline{0.5152 \qquad} \\ & \quad 2.7048 - 1 \\ &= 1.7048 = \log 50.68. \end{aligned}$$

Hence the product is -50.68.

If we multiply in the ordinary way, the result is found to be -50.68128. The result obtained by logarithms is therefore correct to four significant figures.

3. Find the quotient of $17.28 \div 1.44$.

$$\begin{aligned} \log 17.28 &= 1.2375 \\ \log 1.44 &= \underline{0.1584} \\ & \quad 1.0791 = \log 12. \end{aligned}$$

Hence $17.28 \div 1.44 = 12$.

In this case we see that $\log 12 = 1.0792$, while the antilogarithm of 1.0791 is 11.997; that is, to four significant figures 12.00, or 12.

4. Find the quotient of $62.5 \div 0.025$.

$$\begin{aligned}\log 62.5 &= 1.7959\\ \log 0.025 &= \underline{0.3979 - 2}\\ &\ 1.3980 + 2\\ &= 3.3980 = \log 2500, \text{ approximately.}\end{aligned}$$

Hence $62.5 \div 0.025 = 2500$.

5. Find the quotient of $1.457 \div 544.5$.

$$\begin{aligned}\log 1.457 &= 0.1635\\ \log 544.5 &= \underline{2.7360}\end{aligned}$$

Since we cannot subtract the larger logarithm from the smaller, we add an integral number to the characteristic of the logarithm of the dividend and also subtract the same number. In this case we may add and subtract 3. Then we have

$$\begin{aligned}\log 1.457 &= 3.1635 - 3\\ \log 544.5 &= \underline{2.7360}\\ &\ 0.4275 - 3 = \log 0.002676.\end{aligned}$$

Hence $1.457 \div 544.5 = 0.002676$.

6. Find the value of 0.0048^3.

$$\begin{aligned}\log 0.0048 &= 0.6812 - 3\\ &\ \underline{3}\\ &\ 2.0436 - 9\\ &= 0.0436 - 7 = \log 0.0000001106.\end{aligned}$$

The result by actual multiplication is 0.000000110592.

7. Find the value of $\sqrt[7]{2}$.

$$\begin{aligned}\log 2 &= 0.3010.\\ \tfrac{1}{7}\log 2 &= 0.0430 = \log 1.104.\end{aligned}$$

Hence $\sqrt[7]{2} = 1.104$, to four significant places.

8. Find the value of $\sqrt[3]{2.4 \times 3.8 \times 0.0347}$.

$$\begin{aligned}\log 2.4 &= 0.3802\\ \log 3.8 &= 0.5798\\ \log 0.0347 &= \underline{0.5403 - 2}\\ &\ 1.5003 - 2\\ &= 2.5003 - 3\\ \tfrac{1}{3} \text{ of } (2.5003 - 3) &= 0.8334 - 1\\ &= \log 0.6814.\end{aligned}$$

Hence $\sqrt[3]{2.4 \times 3.8 \times 0.0347} = 0.6814$.

It is easier to place the negative characteristic at the right. When we divide by 3 we avoid fractions by writing $2.5003-3$ instead of $1.5003-2$.

Exercise 188. Computations by Logarithms

Perform the following computations by logarithms:

1. 3.67×28.4.
2. 2.57×426.
3. 40.7×90.2.
4. 309×208.
5. 27×4762.
6. 39×289.7.
7. 56×48.92.
8. 73×5.176.
9. 8×0.1728.
10. 9×0.0146.
11. $7.9 \div 6.7$.
12. $15.7 \div 8.3$.
13. $42.8 \div 0.71$.
14. $0.007 \div 0.83$.
15. $0.062 \div 0.09$.
16. $82.83 \div 0.7$.
17. $7.009 \div 9.9$.
18. $7 \div 3.142$.
19. $9 \div 31.47$.
20. $0.6 \div 3.14$.
21. $\sqrt{2}$.
22. $\sqrt[3]{2}$.
23. $\sqrt[4]{7}$.
24. $\sqrt[5]{8}$.
25. $\sqrt[6]{128}$.
26. $\sqrt[7]{147.6}$.
27. $\sqrt[8]{0.0007}$.
28. 42.37^{7}.
29. 5.107^{18}.
30. 0.76^{20}.
31. Find the value of $\sqrt{2.74 \times 42.95 \times 617.8}$.
32. Find the value of $\sqrt[3]{0.7 \times 0.0763 \times 128.4}$.
33. Find the value of $4.76 \times 49.35 \times 72.86 \times 0.07$.

Perform the following multiplications:

34. 4.389×0.000728.
35. 29.76×0.000047.
36. 0.472×0.006234.
37. -29.8×47.63.
38. $-47.82 \times (-2.79)$.
39. $-2.678 \times (-0.0073)$

Perform the following divisions:

40. $\dfrac{27.73}{42.81}$.
41. $\dfrac{0.6987}{3.427}$.
42. $\dfrac{276.9}{0.007342}$.
43. $\dfrac{0.08193}{47.99}$.
44. $\dfrac{0.6398}{0.4926}$.
45. $\dfrac{0.0006872}{0.5283}$.

Perform the following operations:

46. $42^{\frac{2}{3}}$.
47. $368^{\frac{4}{5}}$.
48. $14.92^{\frac{3}{4}}$.
49. $287.9^{\frac{5}{6}}$.
50. $3.142^{\frac{2}{3}}$.
51. $0.0072^{\frac{3}{5}}$.
52. $(-21)^{\frac{2}{3}}$.
53. $(-7.21)^{\frac{3}{5}}$.
54. $(-2.96)^{\frac{1}{7}}$.

332. Cologarithm. The logarithm of the reciprocal of a number is called the *cologarithm* of the number.

The cologarithm of x is expressed thus: $\operatorname{colog} x$.

Since $\operatorname{colog} x = \log \frac{1}{x} = \log 1 - \log x = 0 - \log x$, it is evident that $\operatorname{colog} x = -\log x$.

We may write this $10 - \log x - 10$, thus avoiding negative mantissas. Hence

$$\operatorname{colog} 2 = 10 - \log 2 - 10 = 10 - 0.3010 - 10 = 9.6990 - 10.$$

We may write this $0.6990 - 1$, $\bar{1}.6990$, or $9.6990 - 10$.

333. Use of the Cologarithm. Since to divide by a number is the same as to multiply by its reciprocal, *instead of subtracting the logarithm of a number we may add its cologarithm.*

The cologarithm may be easily written by looking at the logarithm in the table. Thus, since $\log 21 = 1.3222$, we find $\operatorname{colog} 21$ by subtracting this from $10.0000 - 10$, or from $9.999(10) - 10$. That is, beginning at the left we subtract the number represented by each figure from 9, except the right-hand one, which we subtract from 10. In full form we have

$$\begin{array}{rlll}
 & 10.0000 - 10 & = 9.999(10) - 10 & \\
\log 21 = & \underline{1.3222\quad\quad} & = \underline{1.322\ \ 2\quad\quad\ } & \\
\operatorname{colog} 21 = & & \ \ 8.677\ \ 8\ - 10 & = \bar{2}.6778
\end{array}$$

Similarly, we may find colog 0.03952 thus:

$$\begin{array}{rll}
 & 9.999(10) - 10 & \\
\log 0.03952 = & \underline{8.596\ \ 8\ - 10} & \\
\operatorname{colog} 0.03952 = & 1.403\ \ 2 & = 1.4032
\end{array}$$

Practically, of course, we would look at log 0.03952 and subtract mentally, writing down 1.4032 at once.

For example, we see that the cologarithms given below are nearly as easily written as the logarithms:

Logarithms,	3.6042,	5.9605,	7.4316 − 10.
Cologarithms,	6.3958 − 10,	4.0395 − 10,	2.5684.

If we have to subtract a single logarithm, this is simpler than to find the cologarithm and add; but in complicated operations it is easier to use the cologarithm, as is seen on page 398.

Logarithms in which the characteristic is greater than 10 are not common, but in case they occur we may subtract from 11.0000 − 11, 12.0000 − 12, and so on, instead of subtracting from 10.0000 − 10.

334. Advantages of the Cologarithm. The advantages of the cologarithm may be seen in simplifying the following expression:

$$\frac{17.28 \times 6.25 \times 16.9}{1.44 \times 0.25 \times 1.3}.$$

This case has been selected because it is one in which we can easily verify the answer by cancellation. By logarithms we have

$$\begin{aligned}
\log 17.28 &= 1.2375 \\
\log 6.25 &= 0.7959 \\
\log 16.9 &= 1.2279 \\
\text{colog}\, 1.44 &= 9.8416 - 10 \\
\text{colog}\, 0.25 &= 0.6021 \\
\text{colog}\, 1.3 &= \underline{9.8861 - 10} \\
& \ 23.5911 - 20 \\
&= 3.5911 = \log 3900.
\end{aligned}$$

Exercise 189. Computations by Logarithms

Perform the following computations by logarithms:

1. $\dfrac{17.28 \times 1.44}{0.288 \times 8.64}$.

2. $\dfrac{57.5 \times 0.64}{1.25 \times 3.2}$.

3. $\dfrac{12.8 \times 1.341}{14.9 \times 0.64}$.

4. $\sqrt{\dfrac{3 \times 4.1 \times 7.2}{0.2 \times 5.7 \times 9.8}}$.

5. $\sqrt{\dfrac{23 \times 4.8 \times 5.7}{1.9 \times 3.7 \times 0.3}}$.

6. $\sqrt[3]{\dfrac{31 \times 4.8 \times 294.3}{64.8 \times 7.123 \times 4.9}}$.

7. $\sqrt[3]{\dfrac{24 \times 37 \times 428.1}{53.2 \times 41.05 \times 3.7}}$.

8. $\left(\dfrac{0.548 \times 1.98}{39.6 \times 2.74}\right)^{\frac{1}{2}}$.

9. $\left(\dfrac{11.76 \times 1.022}{1.46 \times 39.2}\right)^{\frac{1}{4}}$.

10. $\left(\dfrac{87.68 \times 29.43 \times 51.80}{72.8 \times 0.4 \times 26.43}\right)^{\frac{2}{3}}$.

11. $\sqrt[4]{\dfrac{7.2 \times 3.8 \times 1.46}{1.82 \times 7.46 \times 83.04}}$.

12. $\sqrt[7]{\dfrac{2 \times 4.1 \times 0.7234 \times 96}{1.4 \times 3.82 \times 0.41}}$.

13. $\sqrt[8]{\dfrac{3 \times 42 \times 276 \times 0.2315}{2.7 \times 21.8 \times 375}}$.

14. $\sqrt[10]{\dfrac{1438 \times 2763 \times 1297}{3415 \times 3906 \times 929.8}}$.

APPENDIX

335. Subjects treated. The work set forth in the preceding chapters covers all the topics prescribed in algebra as a preparation for graduation from high school or entrance to college. Occasionally, however, some course of study specifies one or more topics as desirable, although they are not required for the above purposes. To meet demands of this nature these topics are treated in this Appendix. They may be taken, if at all, at the time that the subjects to which they relate are under discussion, or they may be reserved for review in the second year of algebra. The principal topics treated are the following:

1. Detached coefficients, with their application to synthetic multiplication and division. If taken early in the course, the synthetic operations may be omitted unless there is an abundance of time for their treatment.

2. The application of factoring to the solution of equations.

3. The highest common factor and lowest common multiple applied to more difficult cases.

4. A more extended discussion of the theory of fractions than is commonly required.

5. Certain special devices for the solution of simultaneous equations by quadratic methods, formerly more often used than at present.

6. The leading principles of inequalities.

7. Several specimen examination papers and a brief history of algebra are also included. The former will be of service to those who expect to enter college, and it is hoped that the history will be of interest to all.

336. Detached Coefficients in Addition and Subtraction. When we work with compound numbers we frequently write the denomination at the top of a column, and add or subtract the numbers without again writing the denominations until we reach the result. In the same way we may detach the coefficients in algebra, as in the following cases in addition:

$$\begin{array}{cc} \text{ft.} & \text{in.} \\ 2 & 3 \\ 4 & 2 \\ 5 & 4 \\ \hline 11\text{ ft.} & 9\text{ in.} \end{array} \qquad \begin{array}{cc} x & y \\ 2 & +3 \\ 4 & +2 \\ 5 & +4 \\ \hline 11\,x & +\,9\,y \end{array} \qquad \begin{array}{cccccc} x^2 & xy & y^2 & & \text{Check} \\ 5 & +3 & -\ 7 & = & 1 \\ 2 & -8 & +\ 4 & = & -2 \\ 3 & +9 & -\ 8 & = & 4 \\ \hline 10\,x^2 & +\,4\,xy & -\,11\,y^2 & = & 3 \end{array}$$

In the same way we may detach the coefficients in subtraction, as in the following cases:

$$\begin{array}{ccc} x^2 & xy & y^2 \\ 5 & -\ 9 & +\ 8 \\ 3 & +\ 7 & -\ 6 \\ \hline 2\,x^2 & -\,16\,xy & +\,14\,y^2 \end{array} \qquad \begin{array}{cccccc} a^3 & a^2b & ab^2 & b^3 & & \text{Check} \\ 9 & +\ 8 & -7 & +6 & = & 16 \\ 4 & -\ 3 & -9 & +6 & = & -\ 2 \\ \hline 5\,a^3 & +\,11\,a^2b & +\,2\,ab^2 & & = & 18 \end{array}$$

The detaching of coefficients saves time and labor in the writing of algebraic expressions.

Exercise 190. Addition and Subtraction

To be solved by detached coefficients, and checked

1. Add $a^2 - 3\,ab + 7\,b^2$, $8\,a^2 - 2\,ab + 9\,b^2$, $3\,a^2 + 15\,b^2$.
2. Add $x^3 - 3\,x^2y + y^3$, $5\,x^3 + 4\,xy^2 - 7\,y^3$, $x^2y + 9\,xy^2 - 15\,y^3$.
3. Add $m^3 + 5\,m^2n + 10\,mn^2$, $7\,m^2n - n^3$, $4\,m^3 + 3\,mn^2 + n^3$.
4. Add $16\,p^4 + 5\,p^3q - 7\,p^2q^2 + 9\,pq^3 + 10\,q^4$, $8\,p^3q - 3\,pq^3 + 7\,q^4$.
5. Add $21\,x^3 - 17\,x^2y + 15\,xy^2 - 30\,y^3$, $18\,x^2y - 14\,xy^2$, $17\,x^2y$, $31\,y^3 - 20\,x^3$.
6. From $42\,x^2 - 33\,xy + 17\,y^2$ take $28\,x^2 - 43\,xy + 15\,y^2$.
7. From $12\,x^2 + 3\,xy - y^2$ take $11\,x^2 - 35\,xy + 36\,y^2$.
8. From $35\,x^3 - 27\,x^2y + 18\,xy^2 - 7\,y^3$ take $42\,x^2y + xy^2 - 8\,y^3$.

337. Detached Coefficients in Multiplication. In multiplying $x^2 - 3xy + 4y^2$ by $2x - y$ we may detach the coefficients, for we know in advance that the result will begin with $2x^3$ and end with $-4y^3$, and we can easily insert the other letters in descending powers of x and ascending powers of y. Compare the following:

$$\begin{array}{lll}
x^2 - 3xy + 4y^2 & 1 - 3 + 4 & = 2 \\
\underline{2x - y} & \underline{2 - 1} & = \underline{1} \\
2x^3 - 6x^2y + 8xy^2 & 2 - 6 + 8 & \ 2 \\
\underline{\quad - x^2y + 3xy^2 - 4y^3} & \underline{\quad -1 + 3 - 4} & \\
2x^3 - 7x^2y + 11xy^2 - 4y^3 & 2 - 7 + 11 - 4 & = 2
\end{array}$$

Inserting the letters after the several coefficients, the product is $2x^3 - 7x^2y + 11xy^2 - 4y^3$.

The operation by detached coefficients is evidently simpler.

Exercise 191. Multiplication

To be solved by detached coefficients, and checked

1. $(x + 32y)(17x - 5y)$.
2. $(19x + 18y)(3x - 7y)$.
3. $(32a - 17b)(a - b)$.
4. $(26a - 11b)(a + 7b)$.
5. $(7p + 4q)(8p - 9q)$.
6. $(5m + 3n)(7m + 9n)$.
7. $(9m + 7n)(7m + 9n)$.
8. $(31p - 7q)(15p - 8q)$.
9. $(43k + 17)(57k - 19)$.
10. $(62R + 19)(39R - 17)$.
11. $(15F + 9)(17F - 37)$.
12. $(67R - 15)(39R - 19)$.
13. $(x^2 - xy + y^2)(x + y)$.
14. $(x^2 + xy + y^2)(x - y)$.
15. $(3x^2 + xy + 2y^2)(x + y)$.
16. $(5x^2 - xy - y^2)(5x - 7y)$.
17. $(14x^2 + 13x - 1)(7x^2 - 1)$
18. $(15m^2 - 12m + 7)(m - 6)$.
19. $(27p^2 - 13p + 3)(6p + 4)$.
20. $(33p^2 + 15pq - q^2)(p + q)$
21. $(12a^2 - 13ab + b^2)(a + 2b)$
22. $(5K^2 + 3K - 7)(K + 3)$.
23. $(9F^2 + 3F - 1)(F + 2)$.
24. $(R^2 - 7R + 8)(R + 7)$.
25. $(x^3 + 3x^2y + 3xy^2 + y^3)(x^2 + 2xy + y^2)$.
26. $(x^3 - 3x^2y + 3xy^2 - y^3)(x^2 - 2xy + y^2)$.

338. Synthetic Multiplication. When the coefficient of the first term of the multiplier is 1, it is more convenient to write the multiplier at one side, particularly when the coefficients are detached. This arrangement of the polynomials involves no new principle, but the operation, when carried on in this way, is often called *synthetic multiplication.* Compare the following:

$$\begin{array}{rrrr}
x^2 & -\ 7xy & +\ 3y^2 & \\
x & -\ 4y & & \\
\hline
x^3 & -\ 7x^2y & +\ 3xy^2 & \\
 & -\ 4x^2y & +\ 28xy^2 & -\ 12y^3 \\
\hline
x^3 & -\ 11x^2y & +\ 31xy^2 & -\ 12y^3
\end{array}
\qquad
\begin{array}{r|rrrr}
1 & 1 & -7 & +3 & \\
-4 & & -4 & +28 & -12 \\
\hline
 & 1 & -11 & +31 & -12
\end{array}$$

$$x^3 - 11x^2y + 31xy^2 - 12y^3$$

Check. $(-3)(-3) = 9$.

It appears that we need not rewrite the multiplicand when we multiply by 1, and that the whole work by synthetic multiplication is much simpler than by the common method.

Exercise 192. Synthetic Multiplication

To be solved by synthetic multiplication, and checked

1. $(x-3)(9x+17)$.
2. $(x-2)(7x-19)$.
3. $(x+13)(8x+15)$.
4. $(x-19)(9x-13)$.
5. $(a+12)(19a-7)$.
6. $(a-13)(12a-3)$.
7. $(p+q)(27p-39q)$.
8. $(m-3n)(12m+17n)$.
9. $(x+3)(x^2-7x+5)$.
10. $(x+4)(x^2-5x+16)$.
11. $(x-7y)(x^3+4x^2y-3xy^2+7y^3)$.
12. $(a^2-3a+2)(4a^2-7a+16)$.
13. $(a^2-3ab+5b^2)(5a^2+6ab-12b^2)$.
14. $(p^2+5pq-q^2)(p^3+3p^2q-4pq^2+q^3)$.
15. $(m^2-7mn+2n^2)(m^3-3m^2n+9mn^2-7n^3)$.
16. $(x^2+11xy-3y^2)(x^3+2x^2y-3xy^2+5y^3)$.
17. $(x^2-13xy+7y^2)(x^3-3x^2y+4xy^2-3y^3)$.
18. $(a^3+2a^2-3a+1)(a^3-3a^2+4a+2)$.
19. $(a^3+a^2b-2ab^2-b^3)(a^3-a^2b-ab^2+2b^3)$.
20. $(p^3+4p^2-11p+25)(p^3-5p^2+12p-10)$.

339. Detached Coefficients in Division. There is the same gain in detaching coefficients in division as in the other operations, as will be seen from the following, in which $x^3-10x^2+24x-9$ is divided by $x-3$:

$$\begin{array}{r|l} x^3-10x^2+24x-9 & x-3 \\ \cline{2-2} x^3-\ \ 3x^2\qquad\quad & x^2-7x+3 \\ \hline -7x^2+24x\ \ \ & \\ -7x^2+21x\ \ \ & \\ \hline 3x-9 & \\ 3x-9 & \\ \hline \end{array} \qquad \begin{array}{r|l} 1-10+24-9 & 1-3 \\ \cline{2-2} 1-\ \ 3\qquad\quad & 1-7\ +3 \\ \hline -\ \ 7+24\ \ \ & x^2-7x+3 \\ -\ \ 7+21\ \ \ & \\ \hline 3-9 & \\ 3-9 & \\ \hline \end{array}$$

If any power of a letter is missing, write 0 for the coefficient.

340. Synthetic Division. When the divisor is, as in the above case, a binomial with 1 as the coefficient of the first term, the work may be much condensed by the following arrangement:

$$\begin{array}{r|l} & 1-10+24-9 \\ -3 & \quad\ -\ \ 3+21-9 \\ \hline \text{Quotient} = & 1-\ \ 7+\ \ 3;\quad 0 = \text{Remainder.} \end{array}$$

The operation is as follows: 1 is written in the third line, below 1 in the dividend. Then $-3\cdot 1=-3$, and this is taken from -10, leaving -7. Then $-3\cdot(-7)=21$, and this is taken from 24, leaving 3. Then $1-7+3$ is the quotient. Furthermore, $-3\cdot 3=-9$, and this taken from -9 leaves 0, the remainder.

The reasoning is as follows: Since the first term of the divisor is 1, the terms of the quotient will be the same as the first terms of the dividend and the several remainders. By comparing this arrangement with the one in § 339, it is evident that the synthetic operation gives us these first terms, namely, 1, $-10-(-3)=-7$, and $24-21=3$, and that in each case the remainder is $-9-(-9)=0$.

There is an advantage in changing the sign of 3 in the above example, for it permits us to add instead of subtract each time. The usual arrangement is as follows:

$$\begin{array}{r|l} & 1-10+24-9 \\ 3 & \quad\ +\ \ 3-21+9 \\ \hline \text{Quotient} = & 1-\ \ 7+\ \ 3;\quad 0 = \text{Remainder.} \end{array}$$

Exercise 193. Division and Review

Divide by detached coefficients:

1. $x^2 + 2xy + y^2$ by $x + y$.

2. $x^2 - 2xy + y^2$ by $x - y$.

3. $x^2 - y^2$ by $x + y$.

4. $x^3 - y^3$ by $x - y$.

5. $x^3 + 4x^2 + x - 6$ by $x + 2$; by $x + 3$.

6. $x^3 + 4x^2 + x - 6$ by $x^2 + 5x + 6$.

7. $x^4 - 2x^3 - 7x^2 + 8x + 12$ by $x^2 - 2x - 3$.

8. $6x^4 + 7x^3 - 64x^2 + 23x + 28$ by $x^2 + 3x - 4$.

9. $x^5 + 2x^4y + 3x^3y^2 + 4x^2y^3 - xy^4 + y^5$ by $x^2 - 3xy + y^2$.

Divide by synthetic division:

10. $x^4 - 5x^3 + 5x^2 + 5x - 6$ by $x + 1$; by $x - 1$; by $x - 2$.

11. $x^4 - 2x^3 - 7x^2 + 8x + 12$ by $x - 1$; by $x - 2$; by $x - 3$.

12. $x^6 - 14x^4 + 49x^2 - 36$ by $x + 1$; by $x - 1$; by $x - 2$.

13. $a^4 - b^4$ by $a + b$; by $a - b$; by $a - 2b$, and in this case state the quotient and the remainder.

14. Multiply $x^3 - 7x^2 + 4x - 1$ by $x - 7$, using detached coefficients. Check the work by dividing the product by $x - 7$, also using detached coefficients.

15. Divide $2x^3 - x^2 - 12x - 6$ by $2x^2 - 5x - 3$, using detached coefficients. Check the work by multiplying the quotient by $2x^2 - 5x - 3$, also using detached coefficients.

16. Divide the product of $x + 1$ and $x^2 - 12x + 35$ by $x - 7$, and also by $x - 5$, using detached coefficients in all operations.

17. Divide the product of $x + 7$ and $x^2 - 20x + 99$ by $x - 11$, and also by $x - 9$, using detached coefficients in all operations.

18. Divide the sum of $x^3 - 2x^2 + 40x - 50$ and $-15x^2 + 46x - 62$ by $x - 8$, using detached coefficients in all operations.

19. Divide the product of $a^2 + a - 6$ and $a^2 + a - 20$ by $a - 2$. Divide this quotient by $a + 3$. Divide the result by $a - 4$. Use detached coefficients in all operations.

341. Factoring the General Quadratic Trinomial. As stated on page 140, there are several methods of factoring a trinomial of the type $ax^2 + bx + c$. The teacher may substitute one of the following for the trial method already given:

1. *By splitting the middle term.* Since

$$(mx + n)(px + q) = mpx^2 + (np + mq)x + qn,$$

we see that the coefficient of x is the sum of two numbers (np and mq) whose product is the product of the coefficient of x^2 (that is, mp) and the absolute term (that is, qn).

Consider the trinomial $10x^2 - x - 21$.

What are the two numbers whose sum is -1 and whose product is $10(-21)$, or -210? Evidently -15 and 14.

Then
$$\begin{aligned}10x^2 - x - 21 &= 10x^2 + 14x - 15x - 21\\ &= 2x(5x + 7) - 3(5x + 7)\\ &= (2x - 3)(5x + 7).\end{aligned}$$

That is, to factor the type $ax^2 + bx + c$,

Find two numbers whose algebraic sum is b and product ac. Separate b into these parts and factor by grouping.

2. *By making the first term a square.* We proceed as follows:

$$\begin{aligned}10x^2 - x - 21 &= \frac{100x^2 - 10x - 210}{10}\\ &= \frac{(10x)^2 - (10x) - 210}{10}\\ &= \frac{(10x - 15)(10x + 14)}{10}\\ &= \frac{5(2x - 3)\cdot 2(5x + 7)}{10}\\ &= (2x - 3)(5x + 7).\end{aligned}$$

3. *By substitution.* Taking the above example:

Let
$$y = 10x, \text{ or } \tfrac{1}{10}y = x.$$

Then
$$\begin{aligned}10x^2 - x - 21 &= \tfrac{1}{10}y^2 - \tfrac{1}{10}y - 21\\ &= \tfrac{1}{10}(y - 15)(y + 14)\\ &= \tfrac{1}{10}(10x - 15)(10x + 14)\\ &= (2x - 3)(5x + 7).\end{aligned}$$

4. *By formula.* It is possible to factor an expression of the type $ax^2 + bx + c$ by formula, without any trial.

$$ax^2 + bx + c = \frac{a^2x^2 + bax + ac}{a}$$

$$= \frac{(ax)^2 + b(ax) + \frac{b^2}{4} + ac - \frac{b^2}{4}}{a}$$

$$= \frac{\left(ax + \frac{b}{2}\right)^2 - \left(\frac{b^2}{4} - ac\right)}{a}$$

$$= \frac{\left(ax + \frac{b}{2} + \sqrt{\frac{b^2}{4} - ac}\right)\left(ax + \frac{b}{2} - \sqrt{\frac{b^2}{4} - ac}\right)}{a}.$$

For example, consider $10x^2 - x - 21$.

Here $a = 10$, $b = -1$, $c = -21$. Hence the factors are

$$\frac{\left(10x - \frac{1}{2} + \sqrt{\frac{1}{4} + 210}\right)\left(10x - \frac{1}{2} - \sqrt{\frac{1}{4} + 210}\right)}{10}$$

$$= \frac{\left(10x - \frac{1}{2} + \sqrt{\frac{841}{4}}\right)\left(10x - \frac{1}{2} - \sqrt{\frac{841}{4}}\right)}{10}$$

$$= \frac{\left(10x - \frac{1}{2} + \frac{29}{2}\right)\left(10x - \frac{1}{2} - \frac{29}{2}\right)}{10}$$

$$= \frac{(10x + 14)(10x - 15)}{10} = (5x + 7)(2x - 3).$$

5. *By quadratics.* After studying quadratic equations the class may proceed as follows in the factoring of $10x^2 - x - 21$:

Consider what values of x will make $10x^2 - x - 21 = 0$.

If $\qquad 10x^2 - x - 21 = 0,$

then $\qquad x^2 - \frac{1}{10}x + \frac{1}{400} = \frac{21}{10} + \frac{1}{400} = \frac{841}{400}.$

$\qquad \therefore x - \frac{1}{20} = \pm \frac{29}{20},$

and $\qquad x = \frac{1}{20} \pm \frac{29}{20} = \frac{3}{2} \text{ or } -\frac{7}{5}.$

Hence $\qquad x - \frac{3}{2} = 0, \text{ or } x + \frac{7}{5} = 0,$

and $\qquad (x - \frac{3}{2})(x + \frac{7}{5}) = 0,$

or $\qquad (2x - 3)(5x + 7) = 0 = 10x^2 - x - 21.$

An advanced class may study all five of these methods, using the examples on page 142.

342. Equations solved by Factoring. If we have the equation $(x-2)(2x+3)=0$, either one of the factors may equal zero and the equation is thereby satisfied.

For if $x-2=0$, then $0\cdot(2x+3)$ must equal zero;
and if $2x+3=0$, then $(x-2)\cdot 0$ must also equal zero.
But if $x-2=0$, then $x=2$;
and if $2x+3=0$, then $2x=-3$, or $x=-\frac{3}{2}$.

When the second member of an equation is zero, and the first member can be expressed as a product of two or more binomial factors, the equation can be solved by equating each of these factors to zero and solving the resulting simple equations.

1. Solve the equation $x^2-2x-35=0$.

If $x^2-2x-35=0$,
then $(x-7)(x+5)=0$.
Therefore $x-7=0$, and $x=7$,
or $x+5=0$, and $x=-5$.

These results can be checked, for

$$7^2-2\cdot 7-35=0,$$

and $(-5)^2-2\cdot(-5)-35=0$.

2. Solve the equation $x^3-6x^2-19x+84=0$.

Factoring by the Remainder Theorem (§ 117),

$$x^3-6x^2-19x+84=(x-3)(x-7)(x+4)=0.$$

Therefore $x-3=0$, and $x=3$;
or $x-7=0$, and $x=7$;
or $x+4=0$, and $x=-4$.

Therefore the roots are 3, 7, and -4.

3. Solve the equation $x^4-x=0$.

Since $x^4-x=0$,
therefore $x(x-1)(x^2+x+1)=0$.
Therefore $x=0$;
or $x-1=0$, and $x=1$;
or $x^2+x+1=0$.

Solving by quadratics, $x=-\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$.

Hence the four roots are 0, 1, $-\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$.

Exercise 194. Equations solved by Factoring

Examples 1 to 4, oral — Examples 5 to 33, written

1. Solve the equation $(x-3)(x-9)=0$.

2. Solve the equations $(x-3)(x+9)=0$, and $x(x-3)=0$.

3. Solve the equation $(x-2)(x-5)(x-29)=0$.

4. Solve the equation $x(x-2)(x+3)(x-4)=0$.

Solve the following equations:

5. $x^2-7x+10=0$.
6. $x^2-10x+21=0$.
7. $x^2-14x+48=0$.
8. $x^2-7x+12=0$.
9. $x^2-10x+9=0$.
10. $x^2-10x+25=0$.
11. $x^2+4x-21=0$.
12. $x^2+2x-48=0$.
13. $x^2+8x-9=0$.
14. $x^2-25=0$.
15. $x^2+7x+12=0$.
16. $x^2+17x+72=0$.
17. $2x^2-13x-7=0$.
18. $12x^2-25x+12=0$
19. $12x^2+7x-12=0$.
20. $12x^2-7x-12=0$.

21. $x^3-3x^2-x+3=0$; $x^3-7x^2+12x=0$.
22. $x^3-x^2-9x+9=0$; $x^3-10x^2+21x=0$.
23. $x^3-5x^2-4x+20=0$; $x^3+8x^2=9x$.
24. $4x^3-4x^2-x+1=0$; $2x^3-13x^2=7x$.
25. $9x^3-9x^2+4x-4=0$; $12x^3+12x=25x^2$.
26. $16x^4-25x^2+9=0$; $12x^4=7x^3+12x^2$.
27. $36x^4-13x^2+1=0$; $16x^4-17x^2+1=0$.
28. $x^4-10x^3+35x^2-50x+24=0$.
29. $x^4+2x^3-13x^2-14x+24=0$.
30. $(x-\frac{3}{4})(x+\frac{4}{5})(6x^2+27x-15)=0$.
31. $(5x-4)(4x+5)(x^2-6.5x-3.5)=0$.
32. $(x^3-12x^2+27x)(2x^2-21x+27)=0$.
33. $(2x^2+3x)(x^2-22x+57)(x^3+5x^2-84x)=0$.

343. Highest Common Factor. If two polynomials are not readily factored, their highest common factor may be found by another method. This method depends on the principle that

A factor of each of two quantities is a factor of their sum and their difference.

This is apparent because $ax \pm ay = a(x \pm y)$.

Find the H.C.F. of $2x^3 + x^2 - 12x + 9$ and $2x^3 - 7x^2 + 12x - 9$.

The H.C.F. is a factor of the difference of the two expressions, and therefore of $8x^2 - 24x + 18$.

Since 2 is not a factor of either polynomial it cannot be a factor of the H.C.F., and hence we divide by 2 and have $4x^2 - 12x + 9$.

Any factor introduced into but one of the polynomials, if it is not already in the other, will not change the H.C.F., since this is composed of *common* factors only. Therefore we multiply the first polynomial by 2 to avoid a fraction in the quotient.

Since the H.C.F. is a factor of $4x^2 - 12x + 9$, it is a factor of x times $4x^2 - 12x + 9$, or of $4x^3 - 12x^2 + 9x$. The H.C.F. is therefore a factor of the difference, $14x^2 - 33x + 18$. The H.C.F. will not be affected if this is multiplied by 2, because 2 is not a factor of either polynomial. Since the H.C.F. is a factor of $4x^2 - 12x + 9$, it is a factor of 7 times $4x^2 - 12x + 9$, or of $28x^2 - 84x + 63$. The H.C.F. is therefore a factor of the difference, $18x - 27$. Since 9 is not a common factor of the polynomials it may be rejected as a factor of $18x - 27$, and the H.C.F. is therefore a factor of $2x - 3$.

$$
\begin{array}{r|l}
4x^2 - 12x + 9) & 4x^3 + 2x^2 - 24x + 18\,(x \\
 & \underline{4x^3 - 12x^2 + 9x} \\
 & \qquad 14x^2 - 33x + 18 \\
 & \qquad \underline{2} \\
 & \qquad 28x^2 - 66x + 36\,(7 \\
 & \qquad \underline{28x^2 - 84x + 63} \\
 & \qquad\qquad 9)\,\underline{18x - 27} \\
 & \qquad\qquad\quad 2x - 3
\end{array}
$$

But $2x - 3$ is prime; therefore if there is any H.C.F., it is $2x - 3$, and by actual division we show that this is a factor of $4x^2 - 12x - 9$.

It is also a factor of $18x - 27$, and therefore of the sum of $18x - 27$ and $7(4x^2 - 12x + 9)$, or $28x^2 - 66x + 36$. It is a factor of half of this expression, because it does not contain the factor 2. It is therefore a factor of the sum of $14x^2 - 33x + 18$ and of $x(4x^2 - 12x + 9)$, and of half of this sum, that is, of $2x^3 + x^2 - 12x + 9$.

We therefore see that $2x - 3$ is a factor of the two given expressions, and that it is also the highest common factor.

Practically we rarely have to resort to such a long method of finding the H.C.F. Unless the quantities are much larger than usual we can apply the Remainder Theorem to most cases.

1. Find the H.C.F. of

$$6x^3 + x^2 - 5x - 2 \text{ and } 6x^3 + 5x^2 - 3x - 2.$$

Subtract, and the H.C.F. cannot exceed $4x^2 + 2x$. Suppress the factor $2x$, since this is evidently not a common factor of the polynomials. Then the H.C.F. is $2x + 1$ if there is any H.C.F.

$$\begin{array}{r} 6x^3 + 5x^2 - 3x - 2 \\ 6x^3 + x^2 - 5x - 2 \\ \hline 2x\,\big)\,4x^2 + 2x \\ \hline 2x + 1 \end{array}$$

If the expressions are divisible by $2x + 1$, they are divisible by $\frac{1}{2}(2x + 1)$ or by $x + \frac{1}{2}$, or by $x - (-\frac{1}{2})$.

Hence, by the Remainder Theorem, $2x + 1$ is the H.C.F.

2. Find the H.C.F. of $x^4 + x^3 + 2x - 4$ and $x^3 - 2x^2 - 5x + 6$.

By the Remainder Theorem, $x - 1$ is a common factor.

Dividing each by $x - 1$, we have as quotients

$$x^3 + 2x^2 + 2x + 4 = (x + 2)(x^2 + 2),$$

and $$x^2 - x - 6 = (x + 2)(x - 3).$$

Therefore the H.C.F. is $(x - 1)(x + 2)$.

Exercise 195. Highest Common Factor

Find the highest common factor of:

1. $a^3 + 6a^2 - 8a - 7$, and $a^3 + 8a^2 + 10a + 21$.
2. $4m^4 - 5m^2 + 1$, and $4m^4 + 4m^3 + m^2 - 1$.
3. $p^3 + 5p^2 + 8p + 4$, and $p^3 + 6p^2 + 11p + 6$.
4. $6a^3 + 7a^2b - 22ab^2 - 5b^3$, and $15a^3 - 14a^2b - 13ab^2 - 2b^3$.
5. $a^3 - 5a^2b + 9ab^2 - 9b^3$, and $3a^3 - 5a^2b - 7ab^2 - 15b^3$.
6. $6x^3 - 7x^2 - 16x + 12$, and $4x^3 - 8x^2 - 9x + 18$.
7. $4a^4 + 7a^2 + 16$, and $4a^4 + 12a^3 + 9a^2 - 16$.
8. $2m^3 + m^2n - 4mn^2 - 3n^3$, and $2m^3 + m^2n - 9n^3$.
9. $n^3 + 4n^2 + 5n + 2$, and $n^3 + 2n^2 - n - 2$.
10. $6x^3 - 13x^2 + 19x - 7$, and $9x^3 - 27x^2 + 41x - 28$.
11. $2p^3 - 9p^2 + 11p - 3$, and $4p^3 - 4p^2 - 5p + 3$.

344. Lowest Common Multiple. If the expressions are not readily factored, the L.C.M. may be found by using the H.C.F. as an aid in factoring.

Find the L.C.M. of $2x^4 + 3x^3y - 9x^2y^2$, and $6x^4y - 3xy^4 - 17x^3y^2 + 14x^2y^3$.

By § 343, or by using the Remainder Theorem, the H.C.F. is found to be $x(2x - 3y)$. Using this as one factor, we have

$$2x^4 + 3x^3y - 9x^2y^2 = x^2(2x - 3y)(x + 3y).$$

$$\begin{aligned} 6x^4y - 17x^3y^2 + 14x^2y^3 - 3xy^4 &= xy(2x - 3y)(3x^2 - 4xy + y^2) \\ &= xy(2x - 3y)(3x - y)(x - y). \end{aligned}$$

$$\therefore \text{ the L.C.M.} = x^2y(2x - 3y)(x + 3y)(3x - y)(x - y).$$

The L.C.M. of three or more expressions may be obtained by finding the L.C.M. of any two of them; then the L.C.M. of this result and the third expression, and so on.

It is usually as convenient, however, to factor each of the expressions separately and find the L.C.M. in the ordinary manner.

Exercise 196. Lowest Common Multiple

Find the lowest common multiple of:

1. $2x^3 + x^2 - 4x - 3$, and $2x^3 + x^2 - 9$.
2. $a^3 + 4a^2 + 5a + 2$, and $a^3 + 2a^2 - a - 2$.
3. $2x^3 - 9x^2 + 11x - 3$, and $4x^3 - 4x^2 - 5x + 3$.
4. $9a^4 - 4a^2 + 4a - 1$, and $9a^4 - 12a^3 + 4a^2 - 1$.
5. $m^3 - mn^2 + 2m^2n - 2n^3$, and $m^3 - mn^2 - 4m^2n + 4n^3$.
6. $x^3 - x^2y + xy^2 + 14y^3$, and $x^3 - 5x^2y + 13xy^2 - 14y^3$.
7. $9x^2 - 4$, $6x^2 - 13x + 6$, and $6x^2 + 5x - 6$.
8. $x^2 + 2x - 3$, $x^3 + 3x^2 - x - 3$, and $x^3 + 4x^2 + x - 6$.
9. $x^3 + 8x^2 + 19x + 12$, and $x^3 + 9x^2 + 26x + 24$.
10. $6a^3 - 23a^2 + 26a - 8$, and $2a^3 - 13a^2 + 27a - 18$.
11. $5x^4 + 10x^3 + x + 2$, and $10x^4 - 15x^3 + 2x - 3$.
12. $p^4 - p^3 + 3p^2 - 2p + 20$, and $2p^4 + p^3 + 6p^2 - 4p + 16$.
13. $x^6 + 3x^5 + 2x + 6$, and $x^6 + 2x^5 + 2x + 4$.

345. The Laws of Fractions. If desired, the usual laws of fractions may be presented with less regard to arithmetic, as follows:

1. *Multiplying both numerator and denominator of a fraction by the same expression does not change the value of the fraction.*

If $\frac{a}{b} = x,$

then $a = bx,$ Axiom 3

and $na = nbx.$ Axiom 3

Therefore $\frac{na}{nb} = x.$ Axiom 4

But $\frac{a}{b} = x,$

and hence $\frac{a}{b} = \frac{na}{nb}.$ Axiom 6

2. *Dividing both numerator and denominator of a fraction by the same expression does not change the value of the fraction.*

For it has just been proved that

$$\frac{na}{nb} = \frac{a}{b}.$$

3. *Multiplying the numerator by any number multiplies the fraction by that number, and dividing the numerator by any number divides the fraction by that number.*

Let x be the value of the fraction $\frac{a}{b}.$

Then $\frac{a}{b} = x,$

and $a = bx,$

and $na = nbx.$ Axiom 3

Then $\frac{na}{b} = nx.$ Axiom 4

That is, multiplying a by n multiplies x, the value of the fraction, by n.

Likewise, since $a = bx,$

therefore $a \div n = bx \div n,$

and $\frac{a \div n}{b} = \frac{bx \div n}{b} = x \div n.$

That is, dividing a by n divides x, the value of the fraction, by n.

4. *Multiplying the denominator by any number divides the fraction by that number, and dividing the denominator by any number multiplies the fraction by that number.*

Let $\frac{a}{b} = x.$

Then $a = bx,$

and $a = \frac{nbx}{n}.$ Law 1

Therefore $\frac{a}{nb} = \frac{x}{n}.$ Axiom 4

That is, multiplying b by n divides x, the value of the fraction, by n. The proof for division by n is substantially the same.

5. *To multiply one fraction by another, take the product of the numerators for a numerator, and the product of the denominators for a denominator.*

Let the two fractions be $\frac{a}{b}$ and $\frac{m}{n}.$

Let $\frac{a}{b} = x,$ whence $a = bx;$

and let $\frac{m}{n} = y,$ whence $m = ny.$

Then, multiplying, $am = bnxy.$ Axiom 3

Hence, dividing, $\frac{am}{bn} = xy.$ Axiom 4

That is, xy, which is the product of the fractions, is found by taking am for the numerator and bn for the denominator.

6. *To divide one fraction by another, multiply the dividend by the reciprocal of the divisor.*

Let $\frac{a}{b} \div \frac{m}{n} = x.$

Then $\frac{a}{b} = x \cdot \frac{m}{n} = \frac{xm}{n}.$ Axiom 3

Then $\frac{an}{b} = xm,$ Axiom 3

and $\frac{an}{bm} = x.$ Axiom 4

But $\frac{an}{bm}$ is the same as $\frac{a}{b}$ multiplied by the reciprocal of $\frac{m}{n}.$

346. The Forms $\frac{0}{0}$, $\frac{a}{0}$, and $\frac{a}{\infty}$. As stated on page 251, the symbol ∞ is used to mean an infinitely large quantity. The expression $\frac{a}{\infty}$ is read "a divided by infinity." Since we cannot divide by 0, the symbols $\frac{0}{0}$ and $\frac{a}{0}$ have no meaning except as we agree upon some meaning for them.

The symbol $\frac{0}{0}$ means an indeterminate quantity.

For zero multiplied by any finite number, like 2, $\frac{3}{4}$, or -75, is zero. That is, if a stands for any such number, $a \cdot 0 = 0$. If, now, we should consider 0 as a quantity by which we could divide, we should have $a = \frac{0}{0}$.

For this reason it is the custom to look upon $\frac{0}{0}$ as standing for any number, that is, as indeterminate.

The symbol $\frac{a}{0}$ means an infinite number.

For the smaller we make the divisor, the larger the quotient becomes. Therefore the nearer the divisor gets to 0, the larger the quotient becomes. We therefore say that $\frac{a}{0} = \infty$, even though we cannot divide by zero.

In mathematics we must be careful to see that a zero divisor does not enter into our work, since it always leads to an indeterminate or infinite quotient.

For example, let $x = a$.

Then $x^2 = ax$, Axiom 3

and $x^2 - a^2 = ax - a^2$. Axiom 2

Factoring, $(x + a)(x - a) = a(x - a)$.

Dividing by $x - a$, $x + a = a$. Axiom 4

Substituting a for x, $2a = a$.

Dividing by a, $2 = 1$. Axiom 4

The fallacy lies in dividing by $x - a$, which is 0.

The symbol $\frac{a}{\infty}$ means a zero quotient.

For the larger we make the divisor, the smaller the quotient becomes. Therefore when the divisor is infinitely large, the quotient is infinitely small. We therefore say that $\frac{a}{\infty} = 0$, this being a convenient way of saying that as the divisor increases infinitely, the quotient decreases and approaches nearer and nearer zero.

347. Special Devices in Quadratics. In simultaneous quadratics two special devices are sometimes used, one in the case of equations that are homogeneous except for the absolute term (see p. 343), and the other when the two equations are symmetric with respect to x and y; that is, when x and y may be interchanged without altering the equation.

1. Solve the equations

$$x^2 - 3xy + y^2 = -19 \qquad (1)$$

$$x^2 + y^2 = 41 \qquad (2)$$

Let $y = vx$.

Then from (1), $x^2 - 3vx^2 + v^2x^2 = -19$,

whence $x^2 = \dfrac{-19}{1 - 3v + v^2}$.

From (2), $x^2 + v^2x^2 = 41$,

whence $x^2 = \dfrac{41}{1 + v^2}$.

$$\therefore \frac{41}{1 + v^2} = \frac{-19}{1 - 3v + v^2}.$$

Solving for v, $v = \frac{5}{4}$ or $\frac{4}{5}$.

$$\therefore y = \tfrac{5}{4}x \text{ or } \tfrac{4}{5}x.$$

Substituting in (2), $x^2 + \frac{25}{16}x^2 = 41$, or $x^2 + \frac{16}{25}x^2 = 41$.

$$\therefore x = \pm 4, \text{ or } \pm 5.$$

$$\therefore y = \pm 5, \text{ or } \pm 4.$$

2. Solve the symmetric equations

$$x + y = 36 \qquad (1)$$

$$xy = 323 \qquad (2)$$

Let $x = u + v$, and $y = u - v$.

From (1), $2u = 36$, whence $u = 18$.

From (2), $u^2 - v^2 = 323$, whence $18^2 - v^2 = 323$.

Solving, $v = \pm 1$.

Hence $x = u + v = 18 \pm 1 = 19$ or 17,

and $y = u - v = 18 \mp 1 = 17$ or 19.

These two methods may be applied to the appropriate problems on pages 344–352.

348. Inequalities. In elementary algebra but little use is made of inequalities beyond a recognition of the meaning of $a > b$ and $a < b$. The following general principles may, however, be introduced:

1. *If unequals are operated on by positive equals in the same way, the results are unequal in the same order.*

That is, if $x = y$ (both being positive), and $a > b$, then

$$a + x > b + y \qquad ax > by \qquad a^x > b^y$$
$$a - x > b - y \qquad a \div x > b \div y \qquad \sqrt[x]{a} > \sqrt[y]{b}$$

For example, $2 > 1$ and $2 + 5 > 1 + 5$.

2. *If unequals are added to unequals in the same order, the sums are unequal in the same order.*

That is, if $x > y$ and $a > b$, then $x + a > y + b$.

3. *If unequals are subtracted from equals, the remainders are unequal in the reverse order.*

That is, if $x = y$ and $a > b$, then $x - a < y - b$. This is seen in the case of $4 = 4$ and $3 > 2$, where $4 - 3 < 4 - 2$, since $1 < 2$.

The above three principles are used in geometry.

Exercise 197. Inequalities

Illustrate the following statements by using numbers:

1. Positive unequals multiplied by positive unequals in the same order give results unequal in that order.

2. If positive unequals are multiplied by negative equals, the sign of inequality is reversed.

3. The signs of all terms of an inequality may be changed if the inequality sign is reversed.

4. If unequals are subtracted from unequals, the result is indeterminate. It may be an inequality in the same or reversed order, and it may be an equation.

5. Prove that $a^2 + b^2$ is greater than $2ab$ except when $a = b$. Write the general law.

349. Cumulative Review. The following collection of exercises is arranged on a cumulative plan that permits the student, on the completion of each chapter, to review not only the subject just studied, but all of the work that has been covered up to that point. It may be used as circumstances seem to warrant, the appropriate exercise being taken at the close of the several chapters, or a number of exercises being taken at stated periods during the year. In many cases there will be no demand for such a review, but where it is needed it is believed that the problems here given will meet the need, and that the cumulative arrangement will prove helpful.

Exercise 198. Review of Chapter I

1. Define formula, monomial, polynomial, binomial, trinomial, terms of a polynomial, symbols of aggregation, equation, members of an equation, axiom.

2. Give four illustrations of a formula.

3. State the order of operations in an algebraic expression.

4. What is meant by evaluating an expression? Illustrate.

5. Given the formulas $c = \pi d$ and $c = 2\pi r$, derive from them formulas for d and r.

6. If r is the rate for 1 yr., what is the interest on p dollars for 3 mo.? for 6 mo.? for 15 da.? for n years?

7. Given the formula $A = \pi R^2 - \pi r^2$, find the value of A when $R = 14$ and $r = 7$, taking $3\frac{1}{7}$ for the value of π.

Solve the following equations:

8. $2.45 + x = 7.12$.
9. $x - 3.26 = 2.98$.
10. $3x - 1.9 = 3.8$.
11. $7.62 + x = 12$.
12. $2.34x + 2 = 9.02$.
13. $3.19x - 4 = 11.95$.
14. $\frac{2}{3}x = 18$.
15. $\frac{5}{8}x = 3.5$.
16. $\frac{3}{16}x = 7.2$.
17. $\frac{5}{12}x = 0.75$.
18. $\frac{7}{8}x = 0.49$.
19. $\frac{11}{12}x = 1.21$.
20. $x \div 1.9 = 2.4$.
21. $3x \div 7 = 8.1$.
22. $7x \div 3 = 4.9$.
23. $\frac{1}{2}x \div 7 = 9.3$.
24. $2x \div 9 = 8.4$.
25. $2\frac{1}{2}x \div 2 = 12$.

Exercise 199. Review of Chapters I and II

1. An automobile traveling at the rate of d miles an hour goes $5\frac{1}{2}$ mi. in 11 min. Find the value of d.

2. A horse is tethered by a rope 42 ft. long. Over how many square feet can he graze? (Take $3\frac{1}{7}$ for π.)

3. An iron pillar having a circular cross section is 8.4 in. in diameter. What is the area of the cross section?

4. Given the formula $V = e^3$, find the value of V when $e = 7$. Find the value of e when $V = 27$.

5. Solve the equation $17 - x = 14x - 58$.

6. Rule some paper and draw a line showing the growth of the sales of a manufacturer, the sales being as follows: 1910, $100,000; 1911, $140,000; 1912, $180,000; 1913, $230,000; 1914, $290,000; 1915, $370,000; 1916, $480,000.

7. Give four illustrations of a negative number, taken from your experience.

8. Add the following: 7.2, 8.9, −3.4, 6.2, −8.1, 5.3, −13.2.

Subtract as indicated:

9. $3.4 - 1.9$.
10. $1.9 - 3.4$.
11. $2.6 - 5.8$.
12. $5.1 - (-3.1)$.
13. $2.7 - (-4.9)$.
14. $8.6 - (-9.7)$.
15. $-5.3 - 9.8$.
16. $-5.3 - (-9.8)$.
17. $-9.8 - (-5.3)$.

Multiply as indicated:

18. -17×6.3.
19. -9.2×46.
20. $42 \times (-37)$.
21. $5.6 \times (-4.8)$.
22. $-7 \times (-9.3)$.
23. $-9 \times (-8.7)$.

Divide as indicated:

24. $-12.5 \div 2.5$.
25. $-8.68 \div 7$.
26. $12.5 \div (-2.5)$.
27. $8.68 \div (-14)$.
28. $-44.4 \div (-3.7)$.
29. $-58.8 \div (-4.9)$.

30. How much difference in price is there in selling an article at $17.75 below cost and at $24.50 above cost?

Exercise 200. Review of Chapters I–III

1. Define and illustrate factor, literal factor, and numerical factor.

2. Distinguish between coefficient and exponent, and give two illustrations of each.

3. Define and illustrate power and root.

4. Define and illustrate absolute term, similar terms.

5. The terms of a polynomial are $7a^3$, -9, $4a$, and $-2a^2$. Write the polynomial.

6. The factors of a monomial are $7a^3$, -9, $4a$, and $-2a^2$. Write the monomial.

7. If $f(d)$ is πd, what is $f(7)$? $f(14)$? (Take $3\frac{1}{7}$ for π.)

8. Find the fourth root of 256.

9. Write three functions of r that you have studied.

10. If a man is $375 in debt, how will you represent his capital algebraically? How will you represent the capital of a man who is half as much in debt?

Add as indicated:

11. $9.8 + (-7.2)$.
12. $0.9 + (-8.8)$.
13. $-9.7 + 16.9$.
14. $-8.6 + 21.4$.
15. $-7.7 + (-8.9)$.
16. $-5.9 + (-9.5)$.

Subtract as indicated:

17. $16.3 - 7.9$.
18. $16.3-(-7.9)$.
19. $-16.3 - 7.9$.
20. $-6.3 - (-9)$.
21. $-3.1 - (-9.6)$.
22. $-4.2 - (-8.7)$.

Multiply as indicated:

23. $48 \times (-7)$.
24. $(-7) \times (-48)$.
25. $7 \times (-5.9)$.
26. -3.8×6.
27. $-1.9 \times (-4.2)$.
28. $-3\frac{2}{3} \times (-2\frac{1}{3})$.

Divide as indicated:

29. $333 \div (-37)$.
30. $-33.3 \div 3.7$.
31. $-33 \div (-7)$.
32. $-3.3 \div (-9)$.
33. $-\frac{2}{3} \div (-\frac{3}{4})$.
34. $-\frac{3}{4} \div (-\frac{2}{3})$.

Exercise 201. Review of Chapters I–IV

1. Evaluate $3a^2 + 4ab - 7b^2$ for $a = 7$, $b = -2$.

2. Evaluate $4x^3 - 3x^2 + 7x - 7$ for $x = 3$; for $x = -5$.

3. Add 2.8, − 3.2, 4.9, 7.6, − 42.7, and 29.4.

4. From the sum of 19.7 and − 12.9 take − 9.8.

5. Multiply $27.9 + (-13.8)$ by -3; by $-3.1 + 4.1$.

6. Divide $-24.7 + (-28.5)$ by -2; by -4.

7. Given the formula $a = \frac{1}{2}cr$, find the formula for c in terms of a and r.

8. Name the numerical coefficient in $41a^4b^m$; the numerical exponent; the literal exponent. Evaluate the expression for $a = 2$, $b = 5$, $m = 2$.

9. From the sum of 43, − 87, 29, and − 12, subtract the sum of − 19, 4.1, 6.9, − 2, and − 7.

10. Multiply the sum of 21, − 32, 48, and − 56, by the sum of − 3, 4, and − 8.

Add the following expressions:

11.

$$\begin{array}{r} -7a^2 + 3a - 6 \\ 2a^2 - 6a + 9 \\ 8a^2 - 9a + 7 \\ \hline \end{array}$$

12.

$$\begin{array}{r} 4x^2 - 3x + 19 \\ -2x^2 - 7x - 29 \\ 9x^2 + 8x - 32 \\ \hline \end{array}$$

13.

$$\begin{array}{r} 5p^3 + 6p^2 - 7p + 8 \\ 8p^3 - 7p^2 - 9p + 5 \\ 7p^3 - 5p^2 + 8p - 7 \\ 9p^3 + 4p^2 - 8p + 2 \\ \hline \end{array}$$

14.

$$\begin{array}{r} x^4 - 3x^3 + 4x^2 - 7x + 2 \\ 5x^4 - 7x^3 - 3x^2 + 8x - 5 \\ 9x^4 + 8x^3 - 7x^2 + 9x + 7 \\ \hline \end{array}$$

15.

$$\begin{array}{rrrrr} x^4 & & +7x^2 & & -9 \\ & 52x^3 & +3x^2 & -81x & +17 \\ 24x^4 & & -27x^2 & +96x & -13 \\ \hline \end{array}$$

16.

$$\begin{array}{rrrrr} a^4 & -a^3b & +a^2b^2 & -ab^3 & +b^4 \\ a^4 & +2a^3b & -3a^2b^2 & +7ab^3 & -b^4 \\ a^4 & -3a^3b & -7a^2b^2 & +8ab^3 & -b^4 \\ a^4 & +2a^3b & +9a^2b^2 & -14ab^3 & +b^4 \\ \hline \end{array}$$

Exercise 202. Review of Chapters I–V

1. Distinguish between arithmetical sum and algebraic sum.

2. How do you add monomials? Illustrate.

3. Add $x^4 - 3x^2 + 2x - 1$ and $17x^3 - 23x^2 - 48x + 36$.

4. Solve the equation $17x - 23x + 49 = 84 - 11x$.

5. How do you subtract a negative quantity from a positive quantity? Illustrate by the thermometer.

6. How do you subtract a positive quantity from a negative quantity? Illustrate.

7. How do you subtract a negative quantity from a negative quantity? Illustrate.

8. Given the formula $a = \frac{1}{2}bh$, find the formula for h.

Add as indicated:

9. $7\frac{1}{3} + (-5\frac{1}{4})$. 11. $-4\frac{7}{8} + 5\frac{1}{3}$. 13. $-9\frac{3}{5} + (-2\frac{1}{3})$.
10. $8\frac{3}{4} + (-2\frac{1}{2})$. 12. $-6\frac{2}{3} + 9\frac{3}{4}$. 14. $-6\frac{3}{8} + (-3\frac{3}{4})$.

Subtract as indicated:

15. $12\frac{1}{3} - 6\frac{1}{2}$. 17. $-7\frac{1}{2} - 2\frac{3}{8}$. 19. $-5\frac{3}{4} - (-6\frac{1}{3})$.
16. $12\frac{1}{3} - (-6\frac{1}{2})$. 18. $-9\frac{1}{4} - (-1\frac{7}{8})$. 20. $-2\frac{1}{3} - (-4\frac{1}{4})$.

Multiply as indicated:

21. $2\frac{1}{2} \times (-3\frac{1}{4})$. 23. $5 \times (-3\frac{7}{8})$. 25. $-\frac{4}{5} \times (-\frac{3}{8})$.
22. $-2\frac{1}{4} \times 3\frac{1}{2}$. 24. $-4\frac{2}{3} \times 7$. 26. $-\frac{7}{8} \times (-\frac{2}{3})$.

Divide as indicated:

27. $11.9 \div (-7)$. 29. $-\frac{3}{8} \div (-\frac{3}{4})$. 31. $-23.8 \div (-1.7)$.
28. $-1.19 \div 0.7$. 30. $-\frac{7}{8} \div (-\frac{5}{16})$. 32. $-4.76 \div (-1.7)$.

Subtract the following:

33.

$$\begin{array}{r} 21x^3 - 23x^2 + 17x - 13 \\ \underline{18x^3 - 4x^2 - 9x + 27} \end{array}$$

34.

$$\begin{array}{r} 19a^3 + 23a^2 - 42a - 48 \\ \underline{26a^3 - 14a^2 - 12a + 27} \end{array}$$

Exercise 203. Review of Chapters I–VI

1. What is meant by the degree of a term? by the degree of a polynomial? Illustrate each.

2. What is meant by a homogeneous polynomial? Write a homogeneous polynomial of the fourth degree.

3. Write a monomial of the seventh degree involving three letters; of the eighth degree involving the same three letters. Write the product of these two monomials.

4. The area of a circle is πr^2. What is the area of three circles of radius r? Evaluate the result for $\pi = 3\frac{1}{7}$, $r = \frac{7}{66}$.

5. What is the sum of $-17a^2b$, $29a^2b$, and $-8a^2b$? Evaluate the result for $a = -3$, $b = 4$.

6. On a line mark off $+7$ and -4. Find the difference between these two numbers. Show that it may be either positive or negative.

Add the following:

7.

$$\begin{array}{r} 28a^3 - 42a^2 + 27a - 3 \\ 14a^3 + 23a^2 - 36a + 7 \\ -5a^3 - 31a^2 + 46a + 9 \\ \hline \end{array}$$

8.

$$\begin{array}{r} 31x^3 - 33x^2 + 31x - 27 \\ -42x^3 + 48x^2 - 32x + 63 \\ -37x^3 - 21x^2 - 49x - 21 \\ \hline \end{array}$$

Subtract the following:

9.

$$\begin{array}{r} x^3 - 42x^2 + 27x - 3 \\ x^3 - 81x^2 + 13x - 9 \\ \hline \end{array}$$

10.

$$\begin{array}{r} 5x^3 - 4x^2y + 3xy^2 - y^3 \\ 9x^3 - 9x^2y - 3xy^2 + y^3 \\ \hline \end{array}$$

Multiply:

11. $a^3 + 7a^2b - 4ab^2 - 41b^3$ by $a^2 - 2a + 3$.

12. $4m^4 - 3m^2 + 2m - 7$ by $m^2 + 3m - 7$.

13. $5p^4 + 6p^3q - 7pq^3 + 8q^4$ by $p^2 - 2p - q$.

14. $8a^2b^2 + 7a^3b - 3ab^3 + a^4 - b^4$ by $3a^2 - 2ab + 4b^2$.

Exercise 204. Review of Chapters I–VII

1. A steel shaft has a diameter of $3\frac{1}{2}$ in. What is the circumference of the shaft? the area of a cross section?

2. Given the formula $PW = P'W'$, find the value of P when $W = 7$, $P' = 28$, and $W' = 60$.

Solve the following equations:

3. $3x + 4 = x + 38$.

4. $4x - 7 = 2x + 37$.

5. $x + 9 = 7x - 27$.

6. $2x - 3 = 11x - 75$.

7. $\frac{2}{3}x - 3 = x - 7$.

8. $\frac{3}{4}x - 9 = x - 36$.

9. $\frac{1}{3}x + 2 = \frac{2}{3}x - 17$.

10. $\frac{1}{3}x + 5 = \frac{1}{2}x - 12$.

Subtract the following, and check the results:

11.

$$\begin{array}{r} 32x^3 - 4x^2 + 17x - 3 \\ \underline{-9x^3 - 9x^2 + 12x - 8} \end{array}$$

12.

$$\begin{array}{r} 4x^4 - 3x^3 + 2x - 7 \\ \underline{5x^4 - 7x^3 - 3x - 9} \end{array}$$

Multiply the following, and check the results:

13. $a^3 - 3a^2 + 4a + 9$ by $a^2 + 3a - 2$.

14. $p^2 - 4p^3 + 2p - 8$ by $3 + p^2 - 2p$.

15. $x^4 - x^2y^2 + y^4$ by $x^4 + x^2y^2 + y^4$.

16. $(a + b)^2 + 3(a + b) - 4$ by $(a + b)^2 - 1$.

17. $a^5 + 3a^2 - 2a + 3$ by $a^3 - a + 7$.

18. Multiply the product of $a - 7b$ and $8a - b$ by the product of $a + 2b$ and $a - 5b$.

Divide:

19. $x^2 - 7x + 12$ by $x - 3$; by $x - 4$; by $x + 1$.

20. $x^2 + x - 72$ by $x + 9$; by $x - 8$; by $x - 2$.

21. $2x^3 - x^2 + 3x - 9$ by $2x - 3$; by $2x + 3$.

22. $6x^3 + 14x^2 - 4x + 24$ by $3x^2 + 2x + 1$; by $x + 1$.

23. $7x^3 + 58x - 24x^2 - 21$ by $x^2 - 3x + 7$; by $x - 2$.

Exercise 205. Review of Chapters I–VIII

1. Evaluate the formula $a = \frac{1}{2} h(b + b')$ for $h = 27, b = 2\frac{1}{2}$, $b' = 3\frac{3}{4}$; for $h = 32$, $b = 3.5$, $b' = 4.5$.

Evaluate the following for $a = 3$, $b = -2$, $c = 7$:

2. $a^2 - 3b + 9c$.

3. $a^3 - 4b^2 + c^2$.

4. $a^4 - 9b^3 + 2c$.

5. $(a + b)(b + c)$.

6. $(a - 7b)(c - 4b)$.

7. $(a^2 + b^2)(a^2 - b^2)$.

8. $(a - c)(c - 3b)$.

9. $(a^2 - 15b) \div (c + b)$.

Given $A = a^2 - 2a + 3$, $B = a^2 + 7a - 2$, *and* $C = a^2 + 9a - 3$, *find the following:*

10. $A + B$.

11. $A - B$.

12. $A + C$.

13. $B - C$.

14. $A + B + C$.

15. $A + B - C$.

16. $A - B + C$.

17. $A - B - C$.

18. AB.

19. AC.

20. BC.

21. ABC.

Given $P = x^2 - 10x + 21$, $Q = x^2 - 12x + 27$, $R = x - 3$, *find the following:*

22. $P + Q + R$.

23. $P - Q + R$.

24. $Q - P - R$.

25. $Q - 2P + R$.

26. PR.

27. QR.

28. PQ.

29. PQR.

30. $P \div R$.

31. $Q \div R$.

32. $PQ \div R$.

33. $PQ \div R^2$.

Solve the following equations:

34. $9x - 7 = 7x + 35$.

35. $8x - 9 = 4x + 27$.

36. $7x + 8 = 5x + 86$.

37. $2x - 3 = 39 - 5x$.

38. $\frac{2}{3}x = 58$.

39. $\frac{3}{4}x + 9 = 72$.

40. $\frac{4}{5}x - 8 = 86$.

41. $\frac{7}{8}x + 12 = 65$.

42. What number is it that added to 9.07 equals 38.01?

43. If from two thirds of a number we subtract 3 the result is 17. What is the number? Prove it.

Exercise 206. Review of Chapters I–IX

1. What do you mean by a simple equation? Illustrate. What other names are there for a simple equation?

2. What is an identity? Illustrate. Write an equation that is not an identity.

3. When is an equation said to be satisfied? Illustrate.

4. What is meant by solving an equation? How do you determine that a solution is correct?

5. What is meant by the root of an equation? What other meaning has the word "root"?

6. Distinguish between the use of + and − as signs of operation and as signs of quality.

7. What is the difference between the sum of the squares of two quantities and the square of the sum of the quantities?

8. What is the difference between the sum of the squares of two quantities and the square of the difference of the quantities?

9. Show that the difference between the square of the sum of two quantities and the square of the difference of the quantities is four times the product of the quantities.

10. What is the difference between the cube of the sum of two quantities and the cube of the difference of the quantities?

11. What is the quotient of the sum of the fifth powers of two quantities divided by the sum of the quantities?

12. Is the difference of the sixth powers of two quantities always divisible by the difference of the cubes of the quantities? If so, what is the quotient?

13. Is the sum of the sixth powers of two quantities always divisible by the sum of the squares of the quantities? by the sum of the cubes of the quantities? Prove both statements.

14. Is the difference of the eighth powers of two quantities divisible by the difference of the quantities? Prove it.

Multiply:

15. $-x^3 + 2x^2y - y^3$ by $4x^2 + 8xy$.

16. $a^2 + ab + b^2$ by $2a^2 - 3ab - 7b^2$.

17. $x^4 - 3x^3 + 2x^2 - 7x + 1$ by $x^2 + 4x - 3$.

18. $5a^4 - 7a^3b + 4a^2b^2 - 3ab^3 + b^4$ by $a^2 - 3ab + 4b^2$.

19. $7x^4 - 2x^2y^2 - 19xy^3 - 7y^4$ by $x^3 - x^2y + 2xy^2 - y^3$.

Divide:

20. $x^4 - 81y^4$ by $x^3 + 3x^2y + 9xy^2 + 27y^3$.

21. $x^5 - y^5$ by $x^4 + x^3y + x^2y^2 + xy^3 + y^4$.

22. $a^5 + 32b^5$ by $a^4 - 2a^3b + 4a^2b^2 - 8ab^3 + 16b^4$.

23. $2a^4 + 27ab^3 - 81b^4$ by $2a^3 - 6a^2b + 18ab^2 - 27b^3$.

24. $x^4 + 11x^2 - 12x - 5x^3 + 6$ by $3 + x^2 - 3x$.

Solve the following equations:

25. $5x - (3x - 7) = 4x - (6x - 35)$.

26. $6x - 2(9 - 4x) + 3(5x - 7) = 10x - (4 + 16x + 35)$.

27. $9x - 3(5x - 6) + 30 = 0$.

28. $x - 7(4x - 11) = 14(x - 5) - 19(8 - x) - 61$.

29. $(x + 7)(x - 3) = (x - 5)(x - 15)$.

30. To the double of a certain number we add 14 and obtain as a result 154. What is the number?

31. To four times a certain number we add 16 and obtain as a result 188. What is the number?

32. By adding 46 to a certain number we obtain as a result a number three times as large as the original number. Find the original number.

33. One number is three times as large as another. If we take the smaller from 16 and the greater from 30, the remainders are equal. What are the numbers?

34. Divide the number 92 into four parts such that the first exceeds the second by 10, the third by 18, and the fourth by 24.

Exercise 207. Review of Chapters I–X

1. What do you mean by a factor of an algebraic expression? by a monomial factor? by a binomial factor? Illustrate each of these terms.

2. Are the factors of ab the quantities $+a$ and $+b$, or $-a$ and $-b$? Explain your answer.

3. Are the factors of $-x^2+7x-12$ the quantities $x-3$ and $4-x$, or $3-x$ and $x-4$? Verify your answer by multiplying the binomials together.

4. Evaluate the expression $a^3+3a^2b+3ab^2+b^3$ for $a=3$, $b=5$; for $a=4$, $b=-4$.

5. Give illustrations of the following: coefficient, exponent, trinomial, root of an equation, square root of a number, function.

6. Add $7x^5-3x^4+1$, $9x^3-4x^2+7$, $8x^5-4x^4+9$, $-15x^5+7x^4-9x^3+4x-16$.

7. From $x^6-3x^5+7x^4-4x^3+3x^2-9x+36$ subtract $x^6+8x^5-4x^3+5x-96$.

8. Multiply x^4-3x^2+4x-9 by x^3+x^2-3.

9. Divide $x^4-9x^2+x^3-16x-4$ by x^2+4+4x, and check the result.

10. What is the quotient of the sum of the sixth powers of two quantities divided by the sum of their squares?

Solve the following equations:

11. $(x-8)(x+12)=(x+1)(x-6)$.

12. $(x-2)(7-x)+(x-5)(x+3)-2(x-1)+12=0$.

13. $(2x-7)(x+5)=(9-2x)(4-x)+229$.

14. $14-x-5(x-3)(x+2)+(5-x)(4-5x)=45x-76$.

15. $(x+5)^2-(4-x)^2=21x$.

16. $5(x-2)^2+7(x-3)^2=(3x-7)(4x-19)+42$.

17. $(x-3)(x-4)=x(x-1)-30$.

18. The sum of two numbers is 20, and if three times the smaller number is added to five times the larger the sum is 84. What are the numbers?

19. The sum of the ages of a father and son is 80 yr. If the age of the son were doubled, he would be 10 yr. older than the father. What is the age of each?

20. A man has six sons, each 4 yr. older than the next younger. The eldest is three times as old as the youngest. What is the age of each?

21. If we add $24 to a certain sum, the amount will be as much above $80 as the sum is below $80. What is the sum?

22. The sum of $500 is divided among A, B, C, and D. A and B together have $280, A and C $260, and A and D $220. How much has each?

23. If A is twice as old as B, and if he was three times as old as B 22 yr. ago, how old is A?

24. A father is 30 yr. old and his son is 6 yr. old. In how many years will the father be twice as old as the son?

25. A sum of money consists of dollars and quarters, and amounts to $20. There are 50 coins in all. How many are there of each kind?

26. A man paid $15.25 with quarters and half dollars, giving 51 pieces of money in all. How many of each kind were there?

27. A man bought 30 lb. of sugar of two different kinds, paying $1.35 for it all. The better kind cost 5¢ a pound and the poorer kind 3½¢ a pound. How many pounds were there of each kind?

28. Two trains start toward each other at the same time from Buffalo and New York, respectively, 450 mi. apart. The one from New York travels at the rate of 50 mi. an hour, and the other 0.8 as fast. How far from New York will they meet?

Factor the following expressions:

29. $5x^3 - 25x^2$.

30. $16x^4 - 25x^2$.

31. $6x^3 + 18x^2 - 12x$.

32. $49a^2 - 21a + 14$.

33. $y^4 - 2y^3 + y^2$.

34. $45a^9b^7 - 360a^7b^9$.

35. $x^2 - ax - bx + ab$.

36. $ab + ay - by - y^2$.

37. $bc + bx - cx - x^2$.

38. $mx + mn + ax + an$.

39. $cdx^2 - cxy + dxy - y^2$.

40. $ax - ay + by - bx$.

41. The factors of a certain trinomial are $3x^2 - 7$ and $4x^2 + 7$. What is the trinomial?

42. The factors of a certain expression are $a - 7b$ and the square of $a - 7b$. What is the expression? What three prime factors has it?

Factor the following expressions:

43. $x^2 + 36 + 12x$.

44. $x^2 + 196 + 28x$.

45. $x(x + 34) + 289$.

46. $a(a - 8) + 16$.

47. $225 - 30a + a^2$.

48. $x^2 + 361 - 38x$.

49. $z^6 - 34z^3 + 289$.

50. $x^2 + 11x + 24$.

51. $x^2 - 7x + 10$.

52. $a^4 - 1$.

53. $a^8 - 1$.

54. $36x^2 - 49y^2$.

55. $a^4 - 25b^2$.

56. $a^4 - 25b^4$.

57. $(a - b)^2 - c^2$.

58. $x^2 - (a - b)^2$.

59. $12x^2 - 5x - 2$.

60. $12x^2 - 7x + 1$.

61. $289x^2y^2z^2 - 102xy^2z^2d + 9y^2z^2d^2$.

62. $361x^2y^2z^2 - 76abcxyz + 4a^2b^2c^2$.

63. $x^2 + y^2 - z^2 - d^2 - 2xy - 2dz$.

64. Show that one factor of $x^4 + x^2y^2 + y^4$ is $x^2 + xy + y^2$, and find the other factor.

65. Show that one factor of $81x^4 - 34x^2y^2 + y^4$ is $9x^2 + 4xy - y^2$, and find the other factor.

Exercise 208. Review of Chapters I–XI

1. Form an equation whose root is 17 and which contains four terms.

2. What is the sum of $x + x + x + \cdots$ written n times? Evaluate the result for $x = 7\frac{1}{2}$, $n = 17$.

3. If the product of two polynomials is $5x^5 - 3x^4 - x + 1$ and one of them is $1 + 3x^2 - 2x$, what is the other?

4. If the sum of two polynomials is $x^4 - 17x^3 + 14x^2$ and one of them is $27x^3 + 19x^2 - 34$, what is the other?

5. Evaluate $\dfrac{e^c - d^c}{e^2 + ed + d^2}$ for $c = 3$, $d = 4$, $e = 5$.

6. From $2x^2 - 2y^2 - z^2$ take $3y^2 + 2x^2 - z^2$, and from the remainder take $3z^2 - 2y^2 - x^2$.

Simplify the following expressions:

7. $2a - [b - (a - 2b)]$.

8. $3a - [b + (2a - b) - (a - b)]$.

9. $2x + (y - 3z) - [(3x - 2y) + z] + 5x - (4y - 3z)$.

10. $a - [2a + (3a - 4a)] - 5a - \{6a - [(7a + 8a) - 9a]\}$.

Find the product of:

11. $x - 3$, $x - 1$, $x + 1$, and $x + 3$.

12. $x^2 - x + 1$, $x^2 + x + 1$, and $x^4 - x^2 + 1$.

13. $a^2 + ab + b^2$, $a^2 - ab + b^2$, $a^4 - a^2b^2 + b^4$.

14. $4a^3 - 4a^2b + ab^2$, $4a^2 + 3ab + b^2$, and $2a^2b + b^3$.

Divide:

15. $x^4 - 81y^4$ by $x - 3y$.

16. $x^5 - y^5$ by $x - y$.

17. $x^5 + y^5$ by $x + y$.

18. $a^5 + 32b^5$ by $a + 2b$.

19. $a^6 - b^6$ by $a - b$.

20. $a^6 - b^6$ by $a^2 - b^2$.

21. $x^4 + x^3 - 24x^2 - 35x + 57$ by $x^2 + 2x - 3$.

22. $18x^4 + 82x^2 - 67x + 40 - 45x^3$ by $3x^2 - 4x + 5$.

Solve the following equations:

23. $8(10 - x) = 5(x + 3)$.

24. $2x - 3(2x - 3) = 1 - 4(x - 2)$.

25. $(x - 5)(x + 6) = (x - 1)(x - 2)$.

26. $(2x + 3)(3x - 2) = x^2 + x(5x + 3)$.

27. $(5x + 3)(3x + 5) = x(15x + 32) + 37$.

28. $(9x - 7)(3 + 7x) = 3(21x^2 - 7x + 11) - 4x$.

29. Find two numbers differing by 8, such that four times the less exceeds twice the greater by 10.

30. Separate 90 into two parts, such that four times one part equals five times the other.

31. A is twice as old as B, and 20 yr. ago he was three times as old. What is B's age?

Write the product of:

32. $1 + a + b$ and $1 - a - b$.

33. $x + y$, $x - y$, $x^2 + y^2$, and $x^4 + y^4$.

34. $a^2 + ab + b^2$, $a^2 - ab + b^2$, and $a^4 + b^4 + a^2b^2$. Check, letting $a = 1$, $b = 1$.

35. Square $x + y + z$. Check, letting $x = 1$, $y = 2$, $z = 3$.

36. Square $a - b + c - d$. Check, letting $a = 4$, $b = 3$, $c = 2$, $d = 1$.

Write the product of:

37. $(x + 2)(x + 3)$.

38. $(x + 1)(x + 5)$.

39. $(x - 3)(x - 6)$.

40. $(x - 8)(x - 1)$.

41. $(x - 8)(x + 1)$.

42. $(x - 2)(x + 5)$.

43. $(x - 3)(x + 7)$.

44. $(x - 2)(x - 4)$.

45. $(a + 1)(a + 11)$.

46. $(m - 2a)(n + 3a)$.

47. $(p - c)(p - d)$.

48. $(w - 4m)(w + m)$.

Reduce the following to lowest terms:

49. $\frac{x^2-1}{4x(x+1)}$.

50. $\frac{x^2-9x+20}{x^2-7x+12}$.

51. $\frac{x^2-2x-3}{x^2-10x+21}$.

52. $\frac{x^6+2x^3y^3+y^6}{x^6-y^6}$.

53. $\frac{a^3+1}{a^3+2a^2+2a+1}$.

54. $\frac{a^2-a-20}{a^2+a-12}$.

55. $\frac{(a+b)^2}{a^2-ab-2b^2}$.

56. $\frac{a^2+2ab+b^2-c^2}{a^2+ab-ac}$.

Change the following to integral or mixed expressions:

57. $\frac{x^2-2x+1}{x-1}$.

58. $\frac{3x^2+2x+1}{x+4}$.

59. $\frac{3x^2+6x+5}{x+4}$.

60. $\frac{a^2-ax+x^2}{a+x}$.

61. $\frac{2x^2+5}{x-3}$.

62. $\frac{2x^2-5x-2}{x-4}$.

63. $\frac{a^2+b^2}{a-b}$.

64. $\frac{5x^3-x^2+5}{5x^2+4x-1}$.

Change the following to fractional form:

65. $1-\frac{x-y}{x+y}$.

66. $1+\frac{x-y}{x+y}$.

67. $3x-\frac{1+2x^2}{x}$.

68. $a-x+\frac{a^2+x^2}{a-x}$.

69. $\frac{2x^2}{x+y}-(x+y)$.

70. $a-1+\frac{1}{a+1}$.

Add or subtract as indicated:

71. $\frac{1}{x-6}+\frac{1}{x+5}$.

72. $\frac{1}{x-7}-\frac{1}{x-3}$.

73. $\frac{1}{1-x}-\frac{2}{1-x^2}$.

74. $\frac{1}{x-y}+\frac{x}{(x-y)^2}$.

Exercise 209. Review of Chapters I–XII

1. What is the degree of $27\,x^4y^5z^6$ with respect to z? to y and z? to x? to x and y? to x and z? to x, y, and z?

2. Write a homogeneous polynomial of the third degree.

3. By letting $a=2$, $b=3$, and $c=4$, show that $a+(b+c)=(a+b)+c=a+b+c$.

4. Using the values of Ex. 3, show that $a+b+c=a+c+b=c+b+a$.

5. Add $5\,x^4+2\,x^2-7$, $4\,x^3+x-9$, $1+x-x^2$, $x^5+x^4-x^3-x^2-7$, and $9\,x^2+9\,x^3-12\,x-4\,x^4+10$, checking the result by letting $x=1$.

6. Simplify $4\,a-\{3\,a-[2\,a-(a-b)]+5\,b\}$.

7. Multiply $x^4+2\,x^2+4$ by $x^4-2\,x^2+4$, and check.

8. Divide $x^3-5\,ax^2-a^2x+14\,a^3$ by $x^2-3\,ax-7\,a^2$.

9. One factor of $x^3+3\,x^2-13\,x-15$ is $x+1$. Find two other factors.

Factor the following expressions:

10. $x^2+8\,x+7$.

11. $x^2-17\,x+60$.

12. $x^2+7\,x-18$.

13. $x^2-2x-24$.

14. $9\,x^2+30\,x+25$.

15. $16\,x^2-56\,x+49$.

16. x^2+x-72.

17. $x^2-14\,x-176$.

18. $81\,a^4-196\,{}^2b^2$.

19. $729\,a^6-x^6$.

20. $64\,x^7+xy^6$.

21. $(x^2-y^2)^2-y$.

Find the H.C.F. of:

22. $12\,x^2-17\,x+6$ and $9\,x^2+6\,x-8$.

23. x^4-a^4, $x^2+3\,ax-4\,a^2$, and $x^2-5\,ax+4\,a^2$.

Find the L.C.M. of:

24. $x^2-3\,x-4$, x^2-x-12, and $x^2+5\,x+4$.

25. $6\,x^2-13\,x+6$, $6\,x^2+5\,x-6$, and $9\,x^2-4$.

Perform the operations indicated:

26. $\frac{a}{(a+b)b} - \frac{b}{(a-b)a}$.

27. $\frac{5}{2x(x-1)} - \frac{3}{4x(x-2)}$.

28. $\frac{1+x}{1+x+x^2} - \frac{1-x}{1-x+x^2}$.

29. $\frac{x}{x-y} + \frac{x-y}{y-x}$.

30. $\frac{8a^2b^3}{45x^2y} \cdot \frac{15xy^2}{24a^3b^2}$.

31. $\frac{8x^4y}{15ab^3} \div \frac{2x^3}{3ab^2}$.

32. $\frac{9x^2y^2z}{10a^2b^2c} \cdot \frac{-20a^3b^2c}{18xy^2z}$.

33. $\frac{a^3-x^3}{a^3+x^3} \cdot \frac{(a+x)^2}{(a-x)^2}$.

34. $\frac{a^2+b^2}{a^2-b^2} \div \frac{a+b}{a-b}$.

35. $\frac{x^2+xy}{x-y} \div \frac{x^4-y^4}{(x-y)^2}$.

36. $\frac{a^2-4a+3}{a^2-5a+4} \cdot \frac{a^2-9a+20}{a^2-10a+21} \cdot \frac{a^2-7a}{a^2-5a}$.

37. $\frac{b^2-7b+6}{b^2+3b-4} \cdot \frac{b^2+10b+24}{b^2-14b+48} \div \frac{b^2+6b}{b^3-8b^2}$.

Solve the following equations:

38. $(x-3)(x+5) = (x+1)(2x-3) - x^2$.

39. $(x+4)(x-2) = (x+3)(3x+4) - (2x+1)(x-6)$.

40. $(x-3)(2x+5) = x(x+4) + (x+1)(x+3)$.

41. $(x+2)^2 + 3x = (x-2)^2 + 5(16-x)$.

42. The difference between two numbers is 3, and three times the greater number exceeds twice the less by 18. Find the numbers.

Solve the following equations:

43. $5x - \frac{x+2}{2} = 71$.

44. $x - \frac{3-x}{3} = \frac{17}{3}$.

45. $\frac{13-2x}{4} = x - \frac{6x-8}{2}$.

46. $\frac{5x}{2} - \frac{5x}{4} = \frac{9}{4} - \frac{3-x}{2}$.

47. $2x - \frac{5x-4}{6} = 7 - \frac{1-2x}{5}$.

48. $\frac{x+2}{2} = \frac{14}{9} - \frac{3+5x}{4}$.

Solve the following equations:

49. $\frac{5x+3}{8}-\frac{3-4x}{3}+\frac{x}{2}=\frac{31}{2}-\frac{9-5x}{6}.$

50. $\frac{10x+3}{3}-\frac{6x-7}{2}=10(x-1).$

51. $\frac{5x-7}{2}-\frac{2x+7}{3}=3x-14.$

52. $\frac{9x+20}{36}=\frac{4(x-3)}{5x-4}+\frac{x}{4}.$

53. $\frac{9(2x-3)}{14}+\frac{11x-1}{3x+1}=\frac{9x+11}{7}.$

54. $\frac{10x+17}{18}-\frac{12x+2}{13x-16}=\frac{5x-4}{9}.$

55. $\frac{6x+13}{15}-\frac{3x+5}{5x-25}=\frac{2x}{5}.$

56. Find the number whose third and fourth parts added together make 14.

57. Find the number whose third part exceeds its fourth part by 14.

58. The half, fourth, and fifth parts of a certain number are together equal to 76. Find the number.

59. The sum of two numbers is 5760 and their difference is equal to one third of the greater. Find the numbers.

60. The sum of two numbers is 98 and their difference is 36. Find the numbers.

61. The sum of two numbers is s and their difference is d. Find the numbers.

62. Divide 45 into two such parts that the first part divided by 2 shall be equal to the second part multiplied by 2.

63. Find a number such that the sum of its fifth and seventh parts shall exceed the difference of its fourth and seventh parts by 99.

Exercise 210. Review of Chapters I–XIII

1. Evaluate $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ for $x = 2, y = 3$.
2. If $f(x) = x^5 - 5x^4 + 10x^3$, find the value of $f(2)$.
3. Simplify $12x - x - \{7x - [8x - (9x - \overline{3x - 6x})]\}$.

Multiply:

4. $a^2 + ab + b^2$ by $a^2 - ab + b^2$.
5. $a^3 - 3a^2b + 3ab^2 - b^3$ by $a^2 - 2ab + b^2$.
6. $x^2 - xy + y^2 + x + y + 1$ by $x + y - 1$.
7. $4a^7y - 32ay^4 - 8a^5y^2 + 16a^3y^3$ by $a^6y^2 + 4a^2y^4 + 4a^4y^3$.

Find the product of:

8. $a + b$, $a - b$, $a^2 + b^2$, $a^4 + b^4$, and $a^8 + b^8$.
9. $x + a$, $x + 2a$, $x - 3a$, $x - 4a$, and $x + 5a$.
10. $9a^2 + b^2$, $27a^3 - b^3$, $27a^3 + b^3$, and $81a^4 - 9a^2b^2 + b^4$.
11. $a + b - c$, $a + c - b$, $b + c - a$, and $a + b + c$.

Divide:

12. $x^6 - 2x^3 + 1$ by $x^2 - 2x + 1$.
13. $a^4 + 2a^2b^2 + 9b^4$ by $a^2 - 2ab + 3b^2$.
14. $4x^5 - x^3 + 4x$ by $2 + 3x + 2x^2$.
15. $x^4 - 6xy - 9x^2 - y^2$ by $x^2 + y + 3x$.

16. Two casks contain equal quantities of vinegar. From the first cask 34 qt. are drawn, and from the second 20 gal. The quantity now remaining in one cask is twice that remaining in the other. How many quarts did each cask contain at first? How many gallons?

Write the products:

17. $(x - 4y)(x + y)$.
18. $(a - 2b)(a - 5b)$.
19. $(x^2 + 2y^2)(x^2 + y^2)$.
20. $(x + a)(x - b)$.
21. $(ax - 9)(ax + 6)$.
22. $(x^2 - 3xy)(x^2 + xy)$.

Factor the following expressions:

23. $z^4 + 14z^2 + 49$.
24. $a^4 - b^4$.
25. $a^8 - 1$.
26. $y^4 - ay^3 + by^2 + cy$.
27. $x^2 + 3x + 2$.
28. $y^2 - 50yz + 625z^2$.
29. $4x^3y - 12x^2y^2 + 8xy^3$.
30. $cdz^2 - cyz + dyz - y^2$.
31. $(x+1)^2 - (y+1)^2$.
32. $y^2 - 5ay - 50a^2$.
33. $8a^2 + 14ab - 15b^2$.
34. $6a^2 - 19ac + 10c^2$.

35. One factor of $81a^4 - 28a^2b^2 + 16b^4$ is $9a^2 + 10ab + 4b^2$. What is the other factor?

Find the H.C.F. of the following expressions:

36. $6(a-b)^4$, $8(a^2-b^2)^2$, and $10(a^4-b^4)$.
37. $x^2 - y^2$, $(x+y)^2$, and $x^2 + 3xy + 2y^2$.

Find the L.C.M. of the following expressions:

38. $x^2 - 9x - 22$ and $x^2 - 13x + 22$.
39. $x^2 - y^2$, $(x+y)^2$, and $(x-y)^2$.
40. $4ab(a^2 - 3ab + 2b^2)$ and $5a^2(a^2 + ab - 6b^2)$.

Reduce the following fractions to lowest terms:

41. $\dfrac{x^4 + x^2 + 1}{x^2 + x + 1}$.
42. $\dfrac{a^2 + 7a + 10}{a^2 + 5a + 6}$.
43. $\dfrac{a^8 - x^8}{a^2 - x^2}$.
44. $\dfrac{6x^2 - 5x - 6}{8x^2 - 2x - 15}$.

Change the following to fractional form:

45. $a + b - \dfrac{a^2 + b^2}{a + b}$.
46. $7a - \dfrac{2 - 3a + 4a^2}{5 - 6a}$.
47. $3x - \dfrac{5ax - 3}{2a}$.
48. $\dfrac{a+b}{a-b} + 1$.
49. $\dfrac{a-b}{a+b} - 1$.
50. $3x - 10 + \dfrac{41}{x+4}$.

Solve the following equations:

51. $\frac{x}{a}+\frac{y}{b}=c$
$\frac{x}{b}+\frac{y}{a}=-c$

52. $abx+cdy=2$
$ax-cy=\frac{d-b}{bd}$

53. $\frac{x-y+1}{x-y-1}=a$
$\frac{x+y+1}{x+y-1}=b$

54. $(a-b)x=(a+b)y$
$ax-by=\frac{a^2+b^2}{2}$

55. Divide 60 into two parts, such that one part exceeds the other by 24.

56. The sum of two numbers divided by 2 gives as a quotient 24, and their difference divided by 2 gives as a quotient 17. What are the numbers?

57. Three times the greater of two numbers exceeds twice the less by 10, and the sum of twice the greater and three times the less is 24. What are the numbers?

58. A certain fraction equals $\frac{7}{9}$ when the denominator is increased by 4, and equals $\frac{20}{41}$ when the numerator is diminished by 15. What is the fraction?

If $a:b=c:d$, prove that:

59. $a-b:a+b=c-d:c+d$.

60. $ma+nb:ma-nb=mc+nd:mc-nd$.

61. $2a+3b:3a-4b=2c+3d:3c-4d$.

62. $ma^2+nc^2:mb^2+nd^2=a^2:b^2$.

63. A railway passenger observed that a train moving in the opposite direction passed him in 2 sec., but moving in the same direction with him passed him in 30 sec. Compare the rates of the two trains.

64. A certain fraction equals $\frac{2}{3}$ if 7 is added to the numerator, and equals $\frac{3}{8}$ if 7 is subtracted from the denominator. What is the fraction?

Exercise 211. Review of Chapters I–XIV

1. Given the formula $a = \frac{1}{2} h (b + b')$, find the formula for b'. Evaluate the result for $a = 40$, $h = 8$, $b = 6$.

2. Which increases the more rapidly when r increases, the area of a circle or the circumference? Why?

3. Add $4x^6 - 7x^3 + 9x - 19$, $5x^4 - 17x^2 + 3$, $9x^6 - 7x^5 + 4x^4 - 4$, $8x^5 - 5x^4 + 7x^2 - 2$, $12x^3 - 17x + 5$, and $6x^2 - 9x + 7$.

4. From the sum of $x^3 - 9x + 7$ and $x^2 + 7x - 9$ subtract the sum of $x^3 + x^2 - 3$ and $x^2 + 2x - 4$.

5. Multiply $x^3 - x^2 + 3x - 9$ by $x^2 - 7x + 4$ and check the result.

Divide:

6. $x^4 + 64$ by $x^2 + 4x + 8$.

7. $1 - x - 3x^2 - x^5$ by $1 + 2x + x^2$.

8. $x^4 + 9x^2y^2 - 6x^3y - 4y^4$ by $x^2 - 3xy + 2y^2$.

9. $x^5 + x^3 + x^4y + y^3 - 2xy^2 - x^3y^2$ by $x^3 + x - y$.

Write the following expressions in expanded form:

10. $(2x + 1)^2$.

11. $(2a + 5b)^2$.

12. $(3ax - 4x^2)^2$.

13. $(5xy + 2)^2$.

14. $(ab + cd)^2$.

15. $(x - 2y)^3$.

16. $(2x - y)^3$.

17. $(3x + y)^3$.

18. $(x + 3y)^3$.

19. $(2x + 3y)^3$.

Write the quotient of:

20. $\frac{b^3 - 125}{b - 5}$.

21. $\frac{a^3 - 216}{a - 6}$.

22. $\frac{1 + 8a^3}{1 + 2a}$.

23. $\frac{8a^3x^3 + 1}{2ax + 1}$.

24. $\frac{729a^3 + 216b^3}{9a + 6b}$.

25. $\frac{64a^3 + 1000b^3}{4a + 10b}$.

Perform the operations indicated in the following:

26. $\dfrac{x^2+x-2}{x^2-7x}\cdot\dfrac{x^2-13x+42}{x^2+2x}.$

27. $\dfrac{2a(x^2-y^2)^2}{cx}\div\dfrac{(x-y)(x+y)^2}{x^3}.$

28. $\dfrac{x^2-y^2}{x^2-3xy+2y^2}\cdot\dfrac{xy-2y^2}{x^2+xy}\cdot\dfrac{x^2-xy}{(x-y)^2}.$

29. $\dfrac{a^3-3a^2b+3ab^2-b^3}{a^2-b^2}\div\dfrac{2ab-2b^2}{3}\cdot\dfrac{a^2+ab}{a-b}.$

30. $\dfrac{(a+b)^2-c^2}{a^2-(b-c)^2}\div\dfrac{c^2-(a+b)^2}{c^2-(a-b)^2}.$

Solve the following equations:

31. $\dfrac{7x+5}{6}-\dfrac{5x-6}{4}=\dfrac{8-5x}{12}.$

32. $\dfrac{x+4}{3}-\dfrac{x-4}{5}=2+\dfrac{3x-1}{15}.$

33. $\dfrac{9x+20}{36}=\dfrac{4(x-3)}{5x-4}+\dfrac{x}{4}.$

34. $\dfrac{9(2x-3)}{14}+\dfrac{11x-1}{3x+1}=\dfrac{9x+11}{7}.$

35. $\dfrac{x-3}{4(x-1)}=\dfrac{x-5}{6(x-1)}+\dfrac{1}{9}.$

36. A certain fraction equals 2 when 7 is added to its numerator, and equals 1 when 1 is subtracted from its denominator. What is the fraction?

Solve the following equations:

37. $2x+3y=7$
$4x-5y=3$

38. $x-2y=4$
$2x-y=5$

39. $3x-5y=51$
$2x+7y=3$

40. $5x+4y=58$
$3x+7y=67$

Exercise 212. Review of Chapters I–XV

Solve the following equations, plotting each pair:

1. $3x - 4y = -5$
   $4x - 5y = 1$

2. $11x + 3y = 100$
   $4x - 7y = 4$

3. $x + 49y = 50$
   $49x + y = 50$

4. $2x - 7y = 8$
   $4y - 9x = 19$

5. $3x - 4y = 2$
   $7x - 9y = 7$

6. $7x - 5y = 24$
   $4x - 3y = 11$

7. $3x + 2y = 32$
   $20x - 3y = 1$

8. $11x - 7y = 37$
   $8x + 9y = 41$

9. Find the H.C.F. of $x(x+1)^2$, $x^2(x^2-1)$, $2x(x^2-x-2)$.

10. Factor $x^2 - 2xy + y^2 - c^2 + 2cd - d^2$.

11. Simplify $\dfrac{3xy-4}{x^2y^2} - \dfrac{5y^2+7}{xy^3} - \dfrac{6x^2-11}{x^3y}$.

12. Divide $\dfrac{(a+b)^2-(c+d)^2}{(a+c)^2-(b+d)^2}$ by $\dfrac{(a-c)^2-(d-b)^2}{(a-b)^2-(d-c)^2}$.

13. Solve the equation $\dfrac{10x+17}{18} - \dfrac{12x+2}{13x-16} = \dfrac{5x-4}{9}$.

14. A rectangle has its length and breadth respectively 5 ft. longer and 3 ft. shorter than the side of an equivalent square. Find its area.

Solve the following equations:

15. $\dfrac{2}{x+3} = \dfrac{3}{y-2}$
    $\dfrac{x+3}{3} = \dfrac{y-2}{5} + \dfrac{2}{15}$

16. $2x - \dfrac{y-3}{5} = 4$
    $3y + \dfrac{x-2}{3} = 9$

17. $\dfrac{x-4}{5} - \dfrac{y+2}{10} = 0$
    $\dfrac{x}{6} + \dfrac{y-2}{4} = 3$

18. $\dfrac{3x+12y}{11} = 9$
    $\dfrac{1-3x}{7} = \dfrac{11-3y}{5}$

Exercise 213. Review of Chapters 1–XVI

Raise the following expressions to the powers indicated:

1. $(-7\,a^2b^3)^3$.
2. $(24\,a^7b^8)^2$.
3. $(x-2)^4$.
4. $(x+3)^5$.
5. $(abc-4)^3$.
6. $(1-a-a^2)^2$.

Simplify the following expressions:

7. $\sqrt[4]{16\,a^8b^{12}c^{16}}$.
8. $\sqrt[5]{32\,x^{10}y^{15}z^5}$.
9. $\sqrt[3]{-1728\,a^6b^9c}$.
10. $\sqrt[4]{81\,a^8b^{12}c^{16}d}$.
11. $\sqrt[6]{64\,a^6x^{12}}$.
12. $\sqrt[7]{128\,m^7n^{14}p}$.

Find the square root of the following expressions:

13. $x^6-4\,x^5y+8\,x^4y^2-10\,x^3y^3+8\,x^2y^4-4\,xy^5+y^6$.
14. $x^6+25\,x^2+10\,x^4-4\,x^5-20\,x^3+16-24\,x$.
15. Find the square root of 5 to five decimal places.

Find the square root of the following numbers:

16. 120,409.
17. 4816.36.
18. 1867.1041.
19. 1435.6521.
20. 64.128064.
21. 16,803.9369

Simplify the following expressions:

22. $\sqrt{125}$.
23. $\sqrt[3]{162}$.
24. $\sqrt[4]{729}$.
25. $6\sqrt[5]{a^{18}c^8}$.
26. $3\sqrt[6]{a^{12}b^{19}}$.
27. $\sqrt[3]{-1458}$
28. Multiply $2\sqrt{x}-7$ by $3\sqrt{x}$.
29. Divide $\sqrt[3]{2}-\sqrt[3]{6}+\sqrt[3]{10}-\sqrt[3]{12}$ by $\sqrt[3]{2}$.

Divide the following:

30. $\dfrac{3}{\sqrt{7}+\sqrt{5}}$.
31. $\dfrac{7}{2\sqrt{5}-\sqrt{6}}$.
32. $\dfrac{6}{5-2\sqrt{6}}$.
33. $\dfrac{4-\sqrt{2}}{1+\sqrt{2}}$.
34. $\dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$.
35. $\dfrac{7+2\sqrt{10}}{7-2\sqrt{10}}$.
36. Extract the square root of $14+6\sqrt{5}$.
37. Solve the equation $x^2+x-17=x+4\sqrt{15}$.

Exercise 214. Review of Chapters I–XVII

Given $A = x^4 + b^4 - a^2x^2 + 2b^2x$, $B = x^4 - b^4 - 7a^2x^2 - 9b^2x$, $C = x^2 + b^2 + ax$, *perform the operations indicated:*

1. $A + B$.	**3.** $7A - B$.	**5.** AB.	**7.** BC.
2. $A - B$.	**4.** $7B - A$.	**6.** AC.	**8.** $A + C$.

Factor the following expressions:

9. $(a^2 + 2b^2)^2 - a^2b^2$.

10. $(2x - 3y)^2 - (x - 2y)^2$.

11. $15x^2 - 7x - 2$.

12. $11x^2 - 54x + 63$.

Simplify the following expressions:

13. $\dfrac{3x - 2y}{3} - \dfrac{4y + 2x}{5} + \dfrac{22y - 9x}{15}$.

14. $\dfrac{3}{x - 2} + \dfrac{4a}{(x - 2)^2} - \dfrac{5a^2}{(x - a)^3}$.

15. A can do a piece of work in 10 da., and A and B together can do it in 7 da. In how many days can B do it alone?

Find the square root of:

16. 965.9664.

17. $10 + 2\sqrt{21}$.

18. $1 + 4x + 10x^2 + 12x^3 + 9x^4$.

Solve the following equations:

19. $x^2 + 4x = 12$.	**22.** $x^2 - 7x = 8$.	**25.** $x^2 - x = 6$.
20. $x^2 - 6x = 16$.	**23.** $3x^2 - 4x = 7$.	**26.** $5x^2 - 3x = 2$.
21. $x^2 - 12x = -5\frac{3}{4}$.	**24.** $12x^2 + x = 1$.	**27.** $2x^2 - 27x = 14$.

28. $5x(x - 3) - 2(x^2 - 6) = (x + 3)(x + 4)$.

29. Find the radius of a circle whose area would be doubled by increasing its radius 1 in. (Take $3\frac{1}{7}$ for π.)

30. Divide a line 20 in. long into two parts, such that the rectangle contained by the whole line and one part may be equal to the square on the other part.

Exercise 215. Review of Chapters I–XVIII

Solve the following equations:

1. $x^2 - \frac{2}{3}x + \frac{1}{12} = 0.$

2. $\dfrac{3x}{4} + \dfrac{4}{3x} = \dfrac{13}{6}.$

3. $\frac{1}{2}x^2 - \frac{1}{3}x = 2(x + 2).$

4. $\dfrac{x+1}{x+4} = \dfrac{2x-1}{x+6}.$

5. $(x-2)(x-4) - 2(x-1)(x-3) = 0.$

6. $\dfrac{2x+3}{2(2x-1)} - \dfrac{7-x}{2(x+1)} = \dfrac{7-3x}{4-3x}.$

7. $x^2 - 3x - 3 - 6\sqrt{x^2 - 3x - 3} + 5 = 0.$

8. An iron bar weighs 36 lb. If it had been 1 ft. longer, with the same amount of iron, each foot would have weighed $\frac{1}{2}$ lb. less. Find the length of the bar and the weight per foot.

Solve the following equations, plotting each pair:

9. $x + y = 13$
$xy = 36$

10. $x + y = 29$
$xy = 100$

11. $x - y = 19$
$xy = 66$

12. $x - y = 45$
$xy = 250$

13. $x - y = 10$
$x^2 + y^2 = 178$

14. $x - y = 9$
$xy + 8 = 0$

15. $x - y = 1$
$x^2 + y^2 = 8\frac{1}{2}$

16. $5x - 7y = 0$
$20x^2 - 13xy = 16 - 28y^2$

17. $x - y = 7$
$x^2 + xy + y^2 = 13$

18. $x^2 + xy = 35$
$xy - y^2 = 6$

19. $xy - 12 = 0$
$x - 2y = 5$

20. $xy - 7 = 0$
$x^2 + y^2 = 50$

21. $2x - 5y = 9$
$x^2 - xy + y^2 = 7$

22. $x^2 + 4xy = 3$
$4xy + y^2 = 2\frac{1}{4}$

23. $x^2 - xy + y^2 = 48$
$x - y - 8 = 0$

24. $x - 3y = 1$
$xy + y^2 = 5$

Exercise 216. Review of Chapters I–XIX

1. One factor of $64x^4+128x^2y^2+81y^4$ is $8x^2+4xy+9y^2$. What is the other factor?

2. Factor $a^2-2ax+x^2+a-x$.

3. Factor $a^2+2ad+d^2-4b^2+12bc-9c^2$.

4. Simplify $\dfrac{a-b}{(b+c)(c+a)}+\dfrac{b-c}{(c+a)(a+b)}+\dfrac{c-a}{(a+b)(b+c)}$.

5. Solve the equation $\dfrac{6x+13}{15}-\dfrac{3x+5}{5x-25}=\dfrac{2x}{5}$.

Solve the following equations:

6. $$\begin{aligned}5x+3y-6z&=4\\3x-y+2z&=8\\x-2y+2z&=2\end{aligned}$$

7. $$\begin{aligned}x+\frac{y}{2}+\frac{z}{3}&=6\\\frac{x}{3}+y+\frac{z}{2}&=-1\\\frac{x}{2}+\frac{y}{3}+z&=17\end{aligned}$$

8. $$\begin{aligned}4x-5y+2z&=6\\2x+3y-z&=20\\7x-4y+3z&=35\end{aligned}$$

9. $$\begin{aligned}\frac{1}{x}+\frac{2}{y}-5&=0\\\frac{3}{y}-\frac{4}{z}+6&=0\\\frac{4}{x}-\frac{1}{z}-1&=0\end{aligned}$$

10. A sum of money at simple interest amounted to $26,000 in 6 yr., and to $30,000 in 10 yr. Find the sum and the rate.

11. What is the value of $(2^2)^2$? of 2^{2^2}? of $(2^2)^3$? of 2^{2^3}?

Find the cube root of:

12. 274,625.
13. 110,592.
14. 109,215,352.
15. $8x^3+12x^2-30x-35+45x^{-1}+27x^{-2}-27x^{-3}$.

Simplify the following expressions:

16. $\left[(x^{5ab})^3\cdot(x^{5b})^{-2}\right]^{\frac{1}{3a-2}}$.

17. $\left[(a^m)^{m-\frac{1}{m}}\right]^{\frac{1}{m+1}}$.

Exercise 217. Review of Chapters I–XX

1. Define polynomial, and illustrate by three types.

2. The circumference of a circle is a function of what other line? Express your answer by a formula.

3. One factor of $x^4 + 4x^3 - 5x^2 - 36x - 36$ is $x - 3$, and another factor is 5 more than this. Find a third factor.

4. Find the prime factors of $x^6 - 2x^3 + 1$.

5. Find the number whose double increased by 24 exceeds 80 by as much as the number itself is less than 100.

6. What is the square of $x^2 - 5x + 7$?

7. Factor $121x^4 - 286x^2y^2 + 169y^4$.

8. One factor of $25a^4 - 9a^2b^2 + 16b^4$ is $5a^2 + 7ab + 4b^2$. Find the other factor.

Factor the following expressions:

9. $y^2 + 19yz + 48z^2$.
10. $a^4b^6 - 11a^2b^3 + 30$.
11. $c^{10} - 9c^5 - 10$.
12. $6b^2 - 7bx - 3x^2$.
13. $4x^2 + 8x + 3$.
14. $x^4 + x^3 + x^2 + x$.

Find the H.C.F. of the following expressions:

15. $x^2 - y^2$, $x^3 - y^3$, and $x^2 - 7xy + 6y^2$.

16. $x^2 - 1$, $x^3 - 1$, $x^2 + x - 2$, and $2x^2 - 21x + 19$.

Find the L.C.M. of the following expressions:

17. $6(x^2 + xy)$, $8(xy - y^2)$, and $10(x^2 - y^2)$.

18. $x^2 + 11x + 30$, $x^2 + 13x + 42$, and $x^2 + 12x + 35$.

19. Multiply $\dfrac{x^2 - 2xy + y^2 - z^2}{x^2 + 2xy + y^2 - z^2}$ by $\dfrac{x + y - z}{x - y + z}$.

20. Simplify $\dfrac{2}{(x^2 - 1)^2} - \dfrac{1}{2x^2 - 4x + 2} - \dfrac{1}{1 - x^2}$.

21. Solve the equation $\dfrac{6x + 7}{15} - \dfrac{2x - 2}{7x - 6} = \dfrac{2x + 1}{5}$.

Solve the following equations:

22. $8x + 4y - 3z = 6$
$x + 3y - z = 7$
$4x - 5y + 4z = 8$

23. $12x + 5y - 4z = 29$
$13x - 2y + 5z = 58$
$17x - y - z = 15$

24. The sum of the two digits of a number is 10, and if 54 is added to the number the digits will be interchanged. Find the number.

25. Multiply $\sqrt{7} + 3\sqrt{3}$ by $\sqrt{7} - 2\sqrt{3}$.

26. Divide $\sqrt{5} - \sqrt{6}$ by $2\sqrt{5} - \sqrt{6}$.

27. Extract the square root of $9 - 2\sqrt{14}$.

Solve the following equations:

28. $\frac{x^2 - 5}{3} + \frac{2x^2 + 1}{6} = \frac{1}{2}$.

29. $\frac{3}{4x^2} - \frac{1}{6x^2} = \frac{7}{3}$.

30. $\sqrt{x + 1} + \sqrt{x + 16} = \sqrt{x + 25}$.

31. $x^2 + xy + 2y^2 = 74$
$2x^2 + 2xy + y^2 = 73$

32. $x^2 + xy + 4y^2 = 6$
$3x^2 + 8y^2 = 14$

33. The hypotenuse of a right triangle is 20 and the area of the triangle is 96. Find the sides.

If $a : b = c : d$, prove that:

34. $ma : nb = mc : nd$.

35. $(a + 2b) : b = (c + 2d) : d$.

36. Find x when $x + 5 : 2x - 3 = 5x + 1 : 3x - 3$.

37. Find x when $\sqrt{x} + \sqrt{b} : \sqrt{x} - \sqrt{b} = a : b$.

38. Insert ten arithmetical means between -7 and 114.

39. The sum of the first six terms of an arithmetical progression is 27, and the first term is 1. Write the series.

40. If $a = 2$ and $r = 3$, which term of a geometric progression will equal 162?

41. Find the sum of eight terms of a geometric progression whose last term is 1 and fifth term $\frac{1}{8}$.

350. Specimen Examination Papers. The following papers will be found of service in preparing for college entrance examinations. Three hours are allowed on Exercises 218 and 219, and two hours on Exercises 220 and 221.

Exercise 218. Elementary Algebra

Answer eight questions, selecting two from each group

GROUP I

1. Solve $\frac{a+bx}{3b+2ax}=\frac{a-bx}{b-2ax}$.

2. By factoring find the H.C.F. of $27m^5-8m^2$, $6m^3+8m^2-8m$, $12m^4-8m^3$, $27m^3-12m$.

3. Find the product of $2x+3y-z$ and $x-3y+2z$. Check by letting $x=1$, $y=2$, and $z=3$.

GROUP II

4. Reduce each of the following to its simplest form:

$$\sqrt{50}-\sqrt{32},\ 2\sqrt{5}\times\sqrt{15},\ 6\sqrt{20}\div 2\sqrt{10},\ \sqrt{\tfrac{5}{8}},\ \sqrt{9\sqrt{a^4}}.$$

5. Same as page 240, Ex. 7.

6. Find the square root of the following:

$$49x^6-42x^5-47x^4-4x^3+28x^2+16x+4.$$

GROUP III

7. Same as page 244, Ex. 43.

8. Same as page 200, Ex. 52.

9. Solve $\sqrt{2x+1}=2\sqrt{x}-\sqrt{x-3}$.

GROUP IV

10. Same as page 342, Ex. 24.

11. Solve $1-10ax+16a^2x^2=0$.

12. A and B can together address 100 envelopes in an hour; when each works alone A can address 100 envelopes in 50 minutes less time than B. How many can each address in an hour?

Exercise 219. Intermediate Algebra

Answer eight questions, selecting at least two from each group

GROUP I

1. Factor the following: $2a^2-5ab+3ac+2b^2-6bc$; a^4+64b^4; $a^{10}+b^{10}$; $(x-m)^3-8$; $6x^4+2x^2y^2-8y^4$.

2. Solve $\dfrac{\dfrac{x-1}{a}}{\dfrac{a+1}{x}}=\dfrac{\dfrac{x}{a-1}}{\dfrac{a}{a+x}}$.

3. Find the square root of $1+4x$ correct to four places.

4. Same as page 195, Ex. 9.

GROUP II

5. Simplify $\dfrac{2^{-3}\times 3^{-2}\times 4^{-\frac{3}{2}}}{9^{-1}\times 8^{-\frac{5}{3}}}$; $a^{-\frac{1}{3}}\times 2a^{\frac{1}{3}}$; $\dfrac{x^{\frac{1}{2}}\sqrt[6]{x^{-7}}}{x^{-\frac{1}{3}}\sqrt[3]{x^{-5}}}$.

6. Simplify $(-3\sqrt{-3})(5\sqrt{-3})$; $\dfrac{-\sqrt{12}}{\sqrt{-6}}$; $(\sqrt{-1})^5$.

7. Find the ratio of x to y in the following equation:

$$\frac{4x-3y}{4y-3x}=\frac{3x-2y}{4x}.$$

8. Insert 3 geometric means between a and l.

GROUP III

9. Same as page 349, Ex. 44.

10. Plot the graph of $x^2-3x-5=0$ and from the graph find the approximate values of the roots of the equation.

11. Solve $x^{-3}+\dfrac{16}{x^{-3}}=10$.

12. Same as page 333, Ex. 26.

Exercise 220. Algebra to Quadratics

Six questions are required. They must include two questions from Group I, two from Group II, and both questions of Group III.

GROUP I

1. (a) Factor $3 - 192x^6$, $a^2 - 6a - 4b^2 - 12b$, and $6x^2 + x - 15$.

(b) Simplify $\dfrac{2a-b}{x-2a} - \dfrac{b-4a}{2a+x} - \dfrac{3x(b-2a)}{4a^2-x^2}$.

2. (a) Find the value of t from the equation $v = u + ft$; substitute this value in the equation $s = ut + \frac{1}{2}ft^2$ and simplify.

(b) In (a) solve for s when $f = 32$, $v = 5.1$, and $u = 1.3$.

3. The H.C.F. of two given expressions is $a(a-b)$; their L.C.M. is $a^2b(a+b)(a-b)^2$. If one expression is $ab(a^2-b^2)$, what is the other? Prove it.

GROUP II

4. Same as page 201, Ex. 65.

5. If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = \dfrac{g}{h}$, prove that $\dfrac{g}{h} = \dfrac{5a + 3c - 2e}{5b + 3d - 2f}$.

6. A road leads uphill for 3½ miles and then downhill at the same grade for 1¾ miles. A cyclist rides this total distance in 37 minutes and returns over the same road in 29 minutes. What are his rates uphill and downhill?

GROUP III

7. (a) Simplify $\left(\dfrac{m^2p}{64\,m^{-3}p^{\frac{1}{3}}}\right)^{-\frac{1}{3}}$.

(b) If $p = \dfrac{1}{2}\left(\dfrac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}} - \dfrac{x^{-\frac{1}{2}}}{y^{-\frac{1}{2}}}\right)$, show that $(1+p^2)^{\frac{1}{2}} = \dfrac{x+y}{2\sqrt{xy}}$.

8. (a) Arrange $\sqrt{5}$, $\sqrt[4]{24}$, and $\sqrt[3]{11}$ in order of magnitude.

(b) From $\dfrac{3}{5-2\sqrt{3}}$ subtract $\dfrac{1}{2+\sqrt{3}}$ and express the result as a fraction having a rational denominator.

Exercise 221. Quadratics and Beyond

Six questions are required. They must include two questions from Group I, two from Group II, and both questions of Group III.

GROUP I

1. Solve the equation $55x^2 - 73x - 24 = 0$, and the equations $9x^2 + 49y^2 = 85$, $3x - 7y = 13$.

Write the results of the two simultaneous equations so that with each value of x the proper value of y is associated.

2. (a) Solve the equation $9x^2 - 66ax + 149a^2 = 0$.

(b) Having given that the roots of $4x^2 - 2(k+3)x + k^2 = 0$ are equal, find the possible values of k.

3. (a) Solve the equation $\sqrt{2x+5} - \sqrt{6-x} = 1$.

(b) Account for the fact that one of the values obtained for x will not satisfy the given equation.

GROUP II

4. Same as page 201, Ex. 67.

5. Same as page 196, Ex. 10.

6. (a) Draw the graphs of $4y = x^2 + 4$ and $x + 4y = 10$ referred to the same axes.

(b) Use the graphs to estimate the values of x and y which satisfy both equations.

GROUP III

7. (a) Derive a formula for the sum of n terms of an arithmetic progression whose first term is a and common difference d.

(b) Sum the geometric series $3\frac{3}{8} + 2\frac{1}{4} + 1\frac{1}{2} +$ etc. to 6 terms and to infinity.

8. (a) Write the coefficient of x^{15} in the expansion of $x^3(1 - x^2)^{12}$ and simplify it.

(b) Substitute the value unity for x in the expansion of $(1 + x)^n$ and state the resulting theorem relating to the coefficients.

351. History of Algebra. Problems such as we solve by algebra are very old. We find them in a book written in Egypt over 4000 years ago, and afterwards copied by one Ahmes about 1700 B.C. The Ahmes copy, made on a kind of paper called papyrus, is still preserved in the British Museum. It contains problems such as, "Mass, its seventh, its whole, it makes nineteen." We should write this: $\frac{1}{7}x + x = 19$. But Ahmes knew nothing of our algebraic symbols. He had a symbol for "mass," and rude symbols for addition, subtraction, and equality; but an algebraic equation, such as we use, was unknown to the ancient world.

We also find, in a later Egyptian manuscript, a problem that we would solve by quadratics. It is of the form $x + y = a$ and $xy = b$. Nothing but the answer is given, so we do not know how it was solved, but we know that algebra in our sense of the word was not used.

Euclid, who wrote the first great textbook on geometry, and who lived at Alexandria, in Egypt, about 300 B.C., knew and proved $(a + b)^2 = a^2 + 2ab + b^2$, $(a + b)(a - b) = a^2 - b^2$, and similar relations, but all this was proved by geometric figures similar to that on page 111, § 91. This was a cumbersome method, but the ancients were skillful in geometry and so were able to attain results that we now seek by algebra.

Archimedes of Syracuse, in Sicily, about 250 B.C., and others of his time, knew that $c = 2\pi r$ and that $a = \pi r^2$ (§§ 15, 16), but these statements had to be written out in words, the ancients having no good symbols like ours.

The first writer who seems to have developed an algebraic symbolism of any value was Diophantus, who lived in Alexandria about 275 A.D. He had symbols for the unknown quantity, its square, its cube, and so on to the sixth power. He also had a symbol for subtraction and one for equality, and his equations, while written in Greek, were somewhat like ours. He also knew that $-a$ times $-b$ equals $+ab$. He was much interested in indeterminate equations.

It is in the Orient that algebra as we know it had its beginning. The Hindus were much interested in fanciful problems, and some that we still find in our algebras seem to have been first suggested by them. The earliest of these writers was Aryabhatta (about 525 A.D.), who lived at Patna, on the Ganges River. He knew how to solve quadratic equations and had some knowledge of series.

The next great Hindu writer was Brahmagupta (about 650). He lived at Ujjain, formerly a great place for the study of astronomy, in west-central India. He knew how to solve quadratic equations and was also interested in the solution of indeterminate equations.

The third Hindu algebraist of note was Mahavir the Learned (Mahaviracarya), who lived in Mysore, in southern India, about 850. He wrote an extensive treatise on mathematics, which has recently been translated. He had no symbols for operations, and, as was the custom in India, wrote his entire work in verse. The nature of his problems, all of which were rather fanciful, may be inferred from the following: "A pile of apples divided among 2, 3, 4, or 5 persons leaves 1 as a remainder in each case. O you who know arithmetic, tell me the numerical measure of the pile."

The last great Hindu writer, before the European influence began to be manifest, was Bhaskara (about 1150). He had a daughter by the name of Lilavati, and he named his arithmetic after her. His algebra, the Bija Ganita as he called it, contained a number of symbols, but the equations were mostly written out in full. He knew the rules of signs, and that $a \div 0 = \infty$, and could, like his predecessors, solve a quadratic equation.

Meantime the Arabs, chiefly at Bagdad, beginning about 800, made much of algebra. Mohammed ibn Musa al Khowarizmi (Mohammed the son of Moses, the man from Khwarezm — the country about Khiva) wrote a work entitled *Al-jabr wa'l muqābalah* ("reuniting and comparison").

In this the unknown quantity is called the thing, or root. The book contains a very complete discussion of quadratics together with some treatment of surds. It became known in Europe in the Middle Ages, and hence the science went by such names as *Algebra*, *Almucabela*, and *Mucabel*, and the unknown quantity by such names as the Latin *res* ("thing") and the Italian *cosa* ("thing"). From the latter word the science was at one time called the Cossic Art in England and the Coss in Germany.

In eastern Persia the poet Omar Khayyam (about 1050) wrote an algebra in which there is some attempt to solve equations of the third degree of an easy nature, and one of the fourth degree, $(100 - x^2)(10 - x)^2 = 8100$.

One of the early European writers on algebra was Leonardo of Pisa (about 1200), or Leonardo Fibonacci. He wrote on arithmetic, algebra, and geometry, and was able to solve the equation $x^3 + 2x^2 + 10x - 20 = 0$, giving a result equal to 1.3688081075, a remarkable achievement for the time. While yet a boy he was taken by his father to the north coast of Africa, and there studied under a Moorish schoolmaster. At this time the Hindu-Arabic numerals, the ones that we ordinarily use, were known among the Moors but were not yet common in Europe. Upon his return to Italy as a young man Leonardo wrote some books on mathematics and did much to make these numerals better known in Europe. He was one of the best mathematicians of his time.

Our present algebraic symbolism was mostly invented between about 1500 and 1650. The symbols + and − first appeared in print in Johann Widman's arithmetic (1489), the + apparently being suggested by the written form of the Latin *et* ("and"). The Italians at that time commonly used $\bar{p}$ and $\bar{m}$ for these purposes. The first noteworthy printed work to contain algebra was Paciolo's treatise of 1494. He uses *co* (Italian *cosa*, "thing") and ℞ (Latin *res*) for the unknown quantity, *ce* or Z (*census*) for x^2, *cu* or C (*cubus*) for x^3, *ce di ce* (*censo di censo*) for x^4, and so on, and *p* and $\bar{m}$ for plus and

minus. Tartaglia, a great Italian algebraist, who seems to have been the first to solve completely a general equation of the third degree, and who published a well-known treatise in 1556, used *22 men* (*22 men* ℞ *6* for $22 - (22 - \sqrt{6})$, and similar expressions. Cardan, his great rival, published Tartaglia's solution of the cubic equation in 1545. He wrote the equation $x^3 + 6x = 20$ thus: cub^9 p : 6 reb^9 æq̄lis 20. Thus we see that the algebraists of that period had none of the convenient symbolism that we use to-day. For square root Paciolo used ℞ · 2^a, for cube root ℞ · 3^a, and so on. Stifel, a German mathematician of the sixteenth century, edited a work by Christoff Rudolf (1553), and in this he uses our present root signs. The use of x and y for unknown quantities, and $a, b, \cdots,$ for known quantities, is due to the great French mathematician Descartes (1637). To him we also owe our graphs of equations, and he did much, through graphic representations, to make the negative number better understood. Our sign of equality was first used by an English mathematician, Robert Recorde, in 1557. Our symbols of aggregation started with the Italians in the sixteenth century. Bombelli (1572), for example, uses R.q ⌊128. p. 8^2⌋ for $\sqrt{128 + \sqrt{8}}$, where ⌊ stands for *legato* ("bound"); and Cardan uses the expression ℞ v: cu. ℞ 108 p: 10 for $\sqrt{\sqrt[3]{108} + 10}$, the v standing for *universalis*. The symbols $>$ and $<$ were first used by an English algebraist, Harriot, in 1631. This name has particular interest to Americans because Harriot was sent to this country by Sir Walter Raleigh to make a survey of Virginia.

After about 1650 elementary algebra as we know it was substantially complete, except as new applications have come in, and for the past two hundred fifty years it has been the basis of all advanced mathematics. The chief addition to the elementary field has been the better understanding of the complex number, and this is due in large part to De Moivre (1730), Euler (1748), Wessel (1797), and Gauss (about 1830).

352. Table of Powers and Roots. The following table will be found helpful in plotting and solving equations:

No.	Squares	Cubes	Square Roots	Cube Roots	No.	Squares	Cubes	Square Roots	Cube Roots
1	1	1	1.000	1.000	51	2,601	132,651	7.141	3.708
2	4	8	1.414	1.260	52	2,704	140,608	7.211	3.733
3	9	27	1.732	1.442	53	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	54	2,916	157,464	7.348	3.780
5	25	125	2.236	1.710	55	3,025	166,375	7.416	3.803
6	36	216	2.449	1.817	56	3,136	175,616	7.483	3.826
7	49	343	2.646	1.913	57	3,249	185,193	7.550	3.849
8	64	512	2.828	2.000	58	3,364	195,112	7.616	3.871
9	81	729	3.000	2.080	59	3,481	205,379	7.681	3.893
10	100	1,000	3.162	2.154	60	3,600	216,000	7.746	3.915
11	121	1,331	3.317	2.224	61	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	62	3,844	238,328	7.874	3.958
13	169	2,197	3.606	2.351	63	3,969	250,047	7.937	3.979
14	196	2,744	3.742	2.410	64	4,096	262,144	8.000	4.000
15	225	3,375	3.873	2.466	65	4,225	274,625	8.062	4.021
16	256	4,096	4.000	2.520	66	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	67	4,489	300,763	8.185	4.062
18	324	5,832	4.243	2.621	68	4,624	314,432	8.246	4.082
19	361	6,859	4.359	2.668	69	4,761	328,509	8.307	4.102
20	400	8,000	4.472	2.714	70	4,900	343,000	8.367	4.121
21	441	9,261	4.583	2.759	71	5,041	357,911	8.426	4.141
22	484	10,648	4.690	2.802	72	5,184	373,248	8.485	4.160
23	529	12,167	4.796	2.844	73	5,329	389,017	8.544	4.179
24	576	13,824	4.899	2.884	74	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	75	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	76	5,776	438,976	8.718	4.236
27	729	19,683	5.196	3.000	77	5,929	456,533	8.775	4.254
28	784	21,952	5.292	3.037	78	6,084	474,552	8.832	4.273
29	841	24,389	5.385	3.072	79	6,241	493,039	8.888	4.291
30	900	27,000	5.477	3.107	80	6,400	512,000	8.944	4.309
31	961	29,791	5.568	3.141	81	6,561	531,441	9.000	4.327
32	1,024	32,768	5.657	3.175	82	6,724	551,368	9.055	4.344
33	1,089	35,937	5.745	3.208	83	6,889	571,787	9.110	4.362
34	1,156	39,304	5.831	3.240	84	7,056	592,704	9.165	4.380
35	1,225	42,875	5.916	3.271	85	7,225	614,125	9.220	4.397
36	1,296	46,656	6.000	3.302	86	7,396	636,056	9.274	4.414
37	1,369	50,653	6.083	3.332	87	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.362	88	7,744	681,472	9.381	4.448
39	1,521	59,319	6.245	3.391	89	7,921	704,969	9.434	4.465
40	1,600	64,000	6.325	3.420	90	8,100	729,000	9.487	4.481
41	1,681	68,921	6.403	3.448	91	8,281	753,571	9.539	4.498
42	1,764	74,088	6.481	3.476	92	8,464	778,688	9.592	4.514
43	1,849	79,507	6.557	3.503	93	8,649	804,357	9.644	4.531
44	1,936	85,184	6.633	3.530	94	8,836	830,584	9.695	4.547
45	2,025	91,125	6.708	3.557	95	9,025	857,375	9.747	4.563
46	2,116	97,336	6.782	3.583	96	9,216	884,736	9.798	4.579
47	2,209	103,823	6.856	3.609	97	9,409	912,673	9.849	4.595
48	2,304	110,592	6.928	3.634	98	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	99	9,801	970,299	9.950	4.626
50	2,500	125,000	7.071	3.684	100	10,000	1,000,000	10.000	4.642